

ECE531 Spring 2013 Quiz 11

Your Name: SOLUTION

Instructions: This quiz is worth a total of 100 points. The quiz is open book and open notes. You may also use a calculator. You may not use a computer, phone, or tablet. Please show your work on each problem and box/circle your final answers. Points may be deducted for a disorderly presentation of your solution.

Suppose you observe one realization of a random variable $Y \in \mathbb{R}$ drawn from

$$p_Y(y; x) = \frac{y}{x} \exp\left(-\frac{y^2}{2x}\right)$$

with $y \geq 0$ and two hypotheses

$$\mathcal{H}_0 : x = 1$$

$$\mathcal{H}_1 : 1 < x \leq 2.$$

1. 50 points. Show that a uniformly most powerful (UMP) decision rule exists and find the *form* of the UMP decision rule, i.e. find $T(y)$ for

$$\rho^{\text{UMP}}(y) = \begin{cases} 1 & T(y) > v \\ \gamma & T(y) = v \\ 0 & T(y) < v. \end{cases}$$

$T(y)$ should only be a function of y and should be explicit. You do not need to solve for v .

2. 50 points. Now assume the uniform cost assignment (UCA) and the prior

$$\pi(x) = \pi_0 \delta(x - 1) + (1 - \pi_0) \frac{1}{x \ln 2} (u(x - 1) - u(x - 2))$$

where

$$u(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Set up the Bayesian decision rule and simplify as much as possible. You will receive most of the credit for correctly setting up the decision rule. For full credit, you should also be able to compute the integral(s). Hint: $\frac{d}{dx} \exp\left(-\frac{y^2}{2x}\right) = \frac{y^2}{2x^2} \exp\left(-\frac{y^2}{2x}\right)$.

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$$g_0(y, \pi) = \frac{2(1-\pi_0)}{y \ln 2} \int_1^2 \frac{y^2}{2x^2} \exp\left(\frac{-y^2}{2x}\right) dx$$

from hint, we have

$$g_0(y, \pi) = \frac{2(1-\pi_0)}{y \ln 2} \exp\left(\frac{-y^2}{2x}\right) \Big|_1^2$$

$$= \frac{2(1-\pi_0)}{y \ln 2} \left(\exp\left(\frac{-y^2}{4}\right) - \exp\left(\frac{-y^2}{2}\right) \right)$$

So

$$\frac{g_0(y, \pi)}{g_1(y, \pi)} = \frac{\frac{2(1-\pi_0)}{y \ln 2} \left(\exp\left(\frac{-y^2}{4}\right) - \exp\left(\frac{-y^2}{2}\right) \right)}{\pi_0 y \exp\left(\frac{-y^2}{2}\right)} > 1$$

↑
decide H_1

\Leftrightarrow decide H_1 if $\frac{1}{y} \left(\exp\left(\frac{y^2}{4}\right) - 1 \right) > \frac{\pi_0 \ln 2}{2(1-\pi_0)}$

$\Rightarrow \delta^{B\pi}(y) = \begin{cases} \frac{1}{y^2} \left(\exp\left(\frac{y^2}{4}\right) - 1 \right) > \frac{\pi_0 \ln 2}{2(1-\pi_0)} & \Rightarrow \text{decide } H_1 \\ = & H_0 \text{ or } H_1 \\ < & \text{decide } H_0 \end{cases}$