ECE531 Spring 2013 Quiz 12

Your Name: SOLUTION

Instructions: This quiz is worth a total of 100 points. The quiz is open book and open notes. You may also use a calculator. You may not use a computer, phone, or tablet. Please show your work on each problem and box/circle your final answers. Points may be deducted for a disorderly presentation of your solution.

Suppose you wish to detect the presence of a temperature trend in Worcester, e.g. an indicator of global warming, by analyzing average monthly temperature data from Worcester over the last 30 years. A plot of the actual average monthly temperatures in Worcester is shown in Figure 1 below.

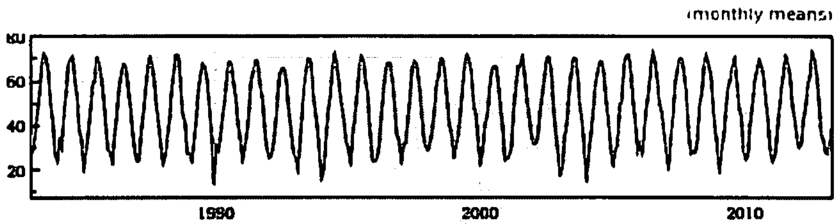

Figure 1: Average monthly temperature in Worcester since 1983 (from Wolfram Alpha).

Each data sample is modeled as

$$Y_k = a + bk + c\sin(\omega_0 k + \phi) + W_k$$

for $k = -n, \dots, n$ corresponding to a month index with ω_0 and ϕ known and $W \sim \mathcal{N}(0, \sigma^2 I)$ with σ^2 known. The parameters a, b, and c are all assumed unknown.

1. 20 points. Briefly explain the physical significance of the parameters a, b, and c.

2. 80 points. Suppose the hypotheses are

$$\begin{aligned} \mathcal{H}_0 &: b = 0 \\ \mathcal{H}_1 &: b \neq 0 \end{aligned}$$

Find the GLRT decision statistic (using approximations as appropriate) and describe how you would find the decision threshold so that the false positive probability constraint $P_{\mathsf{fp}} \le \alpha$ is satisfied. Hint:

$$\text{If } G = \begin{bmatrix} d & 0 & 0 \\ 0 & e & f \\ 0 & f & g \end{bmatrix} \text{ then } G^{-1} = \begin{bmatrix} 1/d & 0 & 0 \\ 0 & g/r & -f/r \\ 0 & -f/r & e/r \end{bmatrix} \text{ with } r = eg - f^2$$

1. a : models the average temperature over the 30 year data record.

b : Models a linear temperature trend over the data

c : Models the effect of seasonality
[Since we the temperatures should be periodic over one year, we would expect $12\,\omega_0 = 2\pi$, so $\omega_0 = \pi/6$]

2. Need to apply theorem 7.1 with $A = [0 \;\; 1 \;\; 0]$ and $b = 0$.

Let $\theta = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ then $H = \begin{bmatrix} 1 & -n & \sin(-n\omega_0 + \phi) \\ 1 & \vdots & \vdots \\ \vdots & \vdots & \\ 1 & n & \sin(n\omega_0 + \phi) \end{bmatrix}$

Then $H^T H = \begin{bmatrix} 2n+1 & 0 & \approx 0 \\ 0 & 2\sum_{k=1}^{n} k^2 & \beta \\ \approx 0 & \beta & n \end{bmatrix}$

since
$$\sum_{k=-n}^{n} \sin^2(k\omega_0 + \phi) = \frac{1}{2}\sum_{k=-n}^{n} \left(1 - \cos(k2\omega_0 + 2\phi)\right)$$
$$\approx \frac{2n+1}{2} \approx n \quad \text{for } n \text{ sufficiently large}$$

and with $\beta = \sum_{k=-n}^{n} k \sin(k\omega_0 + \phi)$ (this is not small, in general)

Hence $(H^T H)^{-1} = \begin{bmatrix} \frac{1}{2n-1} & 0 & 0 \\ 0 & \frac{n}{r} & -\frac{\beta}{r} \\ 0 & -\frac{\beta}{r} & \frac{2\sum_{k=1}^{n} k^2}{r} \end{bmatrix}$ with $r = 2n\sum_{k=1}^{n} k^2 - \beta^2$

continued...

So now apply the theorem...

$$A\hat{\theta}_1 - b \quad A(H^TH)^{-1}H^Ty$$

$$= \begin{bmatrix} 0 & \frac{n}{r} & \frac{-\beta}{r} \end{bmatrix} H^Ty = s^TH^Ty$$

$$A(H^TH)^{-1}A^T = \frac{n}{r}$$

so

$$T(y) = \frac{(s^TH^Ty)^T \frac{r}{n}(s^TH^Ty)}{\sigma^2} = \frac{(\frac{r}{n})\, y^THss^TH^Ty}{\sigma^2}$$

$$ss^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & (\frac{n}{r})^2 & \frac{-\beta n}{r^2} \\ 0 & \frac{-\beta n}{r^2} & \frac{\beta^2}{r^2} \end{bmatrix}$$

$$\frac{r}{n}ss^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{n}{r} & \frac{-\beta}{r} \\ 0 & \frac{-\beta}{r} & \frac{\beta^2}{nr} \end{bmatrix} = \Lambda$$

with $\beta = \sum_{k=-n}^{n} k\sin(k\omega_0 + \phi)$

and $r = 2n\sum_{k=1}^{n} k^2 - \beta^2$

so $\boxed{T(y) = \frac{y^T H \Lambda H^T y}{\sigma^2}}$ is the decision statistic.

Note: the zeros in Λ effectively cause $T(y)$ to ignore the constant component in the observation

To determine the decision threshold, we use the tail probability of a χ^2 random variable with 1 degree of freedom.

But a χ^2 random variable with 1 degree of freedom is equivalent to a squared standard Normal random variable

Hence $T(y) > v \iff |z| > \sqrt{v}$ for $z \sim N(0,1)$

Hence $P_{FP} = 2Q(\sqrt{v}) = \alpha \Rightarrow v = \left(Q^{-1}\left(\frac{\alpha}{2}\right)\right)^2$