ECE531 Spring 2013 Quiz 13

Your Name: SOLUTION

Instructions: This quiz is worth a total of 100 points. The quiz is open book and open notes. You may also use a calculator. You may not use a computer, phone, or tablet. Please show your work on each problem and box/circle your final answers. Points may be deducted for a disorderly presentation of your solution.

1. 40 points. Suppose you have a binary detection problem with hypotheses given as

$$\begin{aligned}\mathcal{H}_0 &: \text{ signal is absent } (Y = W)\\ \mathcal{H}_1 &: \text{ signal is present } (Y = as + W).\end{aligned}$$

where the signal vector s and the scalar amplitude a are both known and $W \sim \mathcal{N}(0, \sigma^2 I)$ with $\sigma^2 > 0$ unknown. In other words, the only unknown parameter here is σ^2. Suppose someone proposes the decision statistic

$$T(y) = \frac{s^\top y}{\sqrt{\frac{1}{n}\|y\|^2}} > v.$$

Is this decision statistic CFAR? Explain.

2. 60 points. In Kay vol. II problem 9.7, Kay derives the GLRT decision statistic (using Kay's notation) as

$$L_G(x) = \frac{p(x; \hat{A}, \hat{\sigma}_1^2, \mathcal{H}_1)}{p(x; \hat{\sigma}_0^2, \mathcal{H}_0)} = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}\right)^{N/2}$$

with

$$\begin{aligned}\hat{\sigma}_0^2 &= \frac{1}{N} x^\top x\\ \hat{\sigma}_1^2 &= \frac{1}{N}(x - H\hat{A})^\top (x - H\hat{A})\\ \hat{A} &= (H^\top H)^{-1} H^\top x \text{ and}\\ H &= [+1, -1, +1, -1, \dots]^\top.\end{aligned}$$

Note that this problem can also be solved with Theorem 9.1. Use Theorem 9.1 to confirm Kay's solution is correct. Hint: Equation (9.13) relates $L_G(x)$ and $T(x)$.

$\boxed{1.}$ To test for CFAR, we can check scale invariance.

$$T(ay) = \frac{a s^T y}{\sqrt{\frac{1}{n} a^2 \|y\|^2}} = \frac{\cancel{a} s^T y}{\cancel{a}\sqrt{\frac{1}{n} s^T y}} = T(y) \checkmark$$

it is ok to assume $a > 0$ here because a would represent the standard deviation of the noise ($a = \sigma$).

$\boxed{2.}$ We will use Kay's notation here...

From (9.13), we have

$$T(x) = \frac{N-p}{r}\left(L_G(x)^{2/N} - 1\right) \qquad p = r = 1$$

$$= (N-1)\left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2} - 1\right) = (N-1)\left(\frac{\hat{\sigma}_0^2 - \hat{\sigma}_1^2}{\hat{\sigma}_1^2}\right)$$

$$= (N-1)\left(\frac{\cancel{\frac{1}{N}} x^T x - \cancel{\frac{1}{N}}(x - H\hat{A})^T(x - H\hat{A})}{\cancel{\frac{1}{N}}(x - H\hat{A})^T(x - H\hat{A})}\right)$$

$$= (N-1)\left(\frac{\cancel{x^T x} - \cancel{x^T x} + 2x^T H\hat{A} - \hat{A} H^T H \hat{A}}{x^T x - 2x^T H\hat{A} + \hat{A} H^T H \hat{A}}\right)$$

Note $\hat{A} = (H^T H)^{-1} H^T x$, so

$$T(x) = (N-1)\left(\frac{2x^T H (H^T H)^{-1} H^T x - x^T H (H^T H)^{-1} \cancel{\boxed{H^T H (H^T H)^{-1}}} H^T x}{x^T x - 2x^T H (H^T H)^{-1} H^T x + x^T H (H^T H)^{-1} \cancel{\boxed{H^T H (H^T H)^{-1}}} H^T x}\right)$$

$$= (N-1)\left(\frac{x^T H (H^T H)^{-1} H^T x}{x^T x - x^T H (H^T H)^{-1} H^T x}\right)$$

$$= (N-1)\left(\frac{x^T P_H x}{x^T (I - P_H) x}\right) = (N-1)\frac{x^T P_H x}{x^T P_H^{\perp} x}$$

which is exactly in the form of theorem 9.1 (as seen in example 9.3) when $A = I$ and $b = 0$.