ECE531 Spring 2013 Quiz 1

Your Name: SOLUTION

Instructions: This quiz is worth a total of 100 points. The quiz is open book and open notes. You may also use a calculator. You may not use a computer, phone, or tablet. Please show your work on each problem and box/circle your final answers. Points may be deducted for a disorderly presentation of your solution.

1. 40 points. Find an unbiased estimator for the unknown scalar parameter $\theta \in \mathbb{R}$ given observations

$$Y_k \overset{\text{i.i.d.}}{\sim} \mathcal{U}(-\theta, \theta).$$

for $k = 0, \ldots, n-1$.

Make a new variable $Z_k = |Y_k|$

Note that $Z_k \overset{\text{i.i.d}}{\sim} U(0, \theta)$

Hence $E\left\{\frac{1}{n} \sum_{k=0}^{n-1} Z_k\right\} = \frac{\theta}{2}$

Then an unbiased estimator for θ could be

$$\boxed{\hat{\theta}(\underline{Y}) = \frac{2}{n} \sum_{k=0}^{n-1} |Y_k|}$$

check unbiased:

$$E\{\hat{\theta}(Y)\} = \frac{2}{n} \sum_{k=0}^{n-1} E\{|Y_k|\} = \frac{2}{n} \cdot n \cdot \frac{\theta}{2} = \theta \quad \checkmark$$

2. 60 points total. Suppose you have an unknown scalar parameter $\theta \in \mathbb{R}$ and get two observations Y_0, Y_1 with the observation model

$$Y_k \overset{\text{i.i.d.}}{=\sim} \mathcal{U}(0, \theta)$$

for $k = 0, 1$. Consider the following two estimators:

$$\hat{\theta}_a(y) = \frac{y_0 + y_1}{\text{[redacted]}}$$

$$\hat{\theta}_b(y) = \frac{3}{2} \max(\{y_0, y_1\})$$

(a) 30 points. Are both estimators unbiased? Hint: The distribution of $Z = \max(\{Y_0, Y_1\})$ is

$$f_Z(z) = \begin{cases} \frac{2z}{\theta^2} & 0 \le z \le \theta \\ 0 & \text{otherwise.} \end{cases}$$

(b) 30 points. Which estimator is better? Explain.

a) $E\{\hat{\theta}_a(Y)\} = E\{Y_0\} + E\{Y_1\} = \frac{\theta}{2} + \frac{\theta}{2} = \theta$ ✓

$E\{\hat{\theta}_b(Y)\} = \frac{3}{2} E\{\max\{Y_0, Y_1\}\} = \frac{3}{2} \int_0^\theta z \frac{2z}{\theta^2} \, dz$

$= \frac{6}{2\theta^2} \left[\frac{z^3}{3}\right]_0^\theta = \frac{6\theta^3}{6\theta^2} = \theta$ ✓

b) Need to compute variances to answer this question...
(since both are unbiased)

$\text{var}\{\hat{\theta}_a(Y)\} = 2\,\text{var}\{Y_0\} = 2 \frac{(\theta - 0)^2}{12} = \frac{2\theta^2}{12} = \boxed{\frac{\theta^2}{6}}$

$\text{var}\{\hat{\theta}_b(Y)\} = \int_0^\theta \left(\frac{3}{2}z - \theta\right)^2 \frac{2z}{\theta^2} \, dz$

$= \frac{2}{\theta^2} \int_0^\theta \left(\frac{9}{4}z^2 - \frac{6}{2}\theta z + \theta^2\right) z \, dz$

$= \frac{2}{\theta^2} \left[\frac{9}{16}\theta^4 - \frac{6}{6}\theta^4 + \frac{\theta^4}{2}\right]$

$= 2\theta^2 \left[\frac{27}{48} - \frac{48}{48} + \frac{24}{48}\right] = \frac{6}{48}\theta^2 = \boxed{\frac{\theta^2}{8}}$

this one has smaller variance
⇒ better.