ECE531 Spring 2013 Quiz 2

Your Name: SOLUTION

Instructions: This quiz is worth a total of 100 points. The quiz is open book and open notes. You may also use a calculator. You may not use a computer, phone, or tablet. Please show your work on each problem and box/circle your final answers. Points may be deducted for a disorderly presentation of your solution.

1. 50 points. Suppose you have a coin that comes up heads with probability $0 \leq \theta \leq 1$. You toss a coin n independent times and define the scalar observation $Y \in \{0, \ldots, n\}$ as the number of heads observed. You wish to estimate the unknown parameter θ, i.e. the probability that the coin comes up heads, from this scalar observation.

   We know that Y is binomially distributed with

   $$p_Y(y;\theta) = \text{Prob}(Y = y; \theta) = \binom{n}{k} \theta^y (1-\theta)^{(n-y)}$$

   for $y \in \{0, \ldots, n\}$ with $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. We also know $\text{E}[Y] = n\theta$ and $\text{var}[Y] = n\theta(1-\theta)$.

   Question: Is $\hat{\theta}(y) = \frac{y}{n}$ an MVU estimator?

2. 50 points. Suppose you receive three observations

   $$Y_k = \cos(\omega k + \phi) + W_k$$

   for $k = -1, 0, 1$ with $W_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2)$. Defining the unknown parameter as $\theta = [\omega, \phi]^\top$, compute the Fisher information matrix. Your final answer should be exact. Do not use any approximations.

$\boxed{1.}$ To answer this question, we can calculate

$$E\{\hat{\theta}(Y)\} = \frac{E\{Y\}}{n} = \frac{n\theta}{n} = \theta \checkmark \quad \text{unbiased}$$

$$\text{var}\{\hat{\theta}(Y)\} = \frac{1}{n^2}\text{var}\{Y\} = \frac{n\theta(1-\theta)}{n^2} = \frac{\theta(1-\theta)}{n}$$

can we use the CRLB?

check: $E\left\{\frac{\partial}{\partial\theta} \ln(P_Y(Y;\theta))\right\} = 0$?

$$\frac{\partial}{\partial\theta} \ln P_Y(Y;\theta) = \frac{\partial}{\partial\theta}\left(\text{constant} + Y\ln(\theta) + (n-Y)\ln(1-\theta)\right)$$

$$= \frac{Y}{\theta} - \frac{n-Y}{1-\theta} = \frac{Y(1-\theta) - (n-Y)\theta}{\theta(1-\theta)}$$

$$= \frac{Y - n\theta}{\theta(1-\theta)}$$

$$E\left\{\frac{Y-n\theta}{\theta(1-\theta)}\right\} = \frac{E\{Y\} - n\theta}{\theta(1-\theta)} = \frac{n\theta - n\theta}{\theta(1-\theta)} = 0$$

so, ok, the condition is satisfied and we can use the CRLB.

$$I(\theta) = E\left\{\left(\frac{\partial}{\partial\theta}\ln P_Y(Y;\theta)\right)^2\right\} = E\left\{\left(\frac{Y-n\theta}{\theta(1-\theta)}\right)^2\right\}$$

$$= \frac{1}{\theta^2(1-\theta)^2} E\{(Y-n\theta)^2\}$$

$$= \frac{1}{\theta^2(1-\theta)^2}\text{var}\{Y\} = \frac{n\theta(1-\theta)}{\theta^2(1-\theta)^2} = \frac{n}{\theta(1-\theta)}$$

Hence $\text{Var}\{\hat{\theta}(Y)\} \geq \frac{\theta(1-\theta)}{n}$

But our estimator achieves this bound, hence it is efficient and must be MVU.

$\boxed{2.}$ Since we have AWGN in this problem, we can use (3.33).

$$I_{ij}(\theta) = \frac{\sum_{k=-1}^{1}\left(\frac{\partial s[k;\theta]}{\partial \theta_i}\right)\left(\frac{\partial s[k;\theta]}{\partial \theta_j}\right)}{\sigma^2}$$

$$s[k;\theta] = \cos(\omega k+\phi) \qquad \text{with} \quad \theta = \begin{bmatrix} \omega \\ \phi \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

$$\frac{\partial s[k;\theta]}{\partial \theta_1} = -k\sin(\omega k+\phi)$$

$$\frac{\partial s[k;\theta]}{\partial \theta_2} = -\sin(\omega k+\phi)$$

$$I_{11}(\theta) = \frac{1}{\sigma^2}\sum_{k=-1}^{1} k^2\sin^2(\omega k+\phi) = \frac{1}{\sigma^2}\left(\sin^2(\omega+\phi)+\sin^2(-\omega+\phi)\right)$$

$$I_{12}(\theta) = \frac{1}{\sigma^2}\sum_{k=-1}^{1} k\sin^2(\omega k+\phi) = \frac{1}{\sigma^2}\left(-\sin^2(-\omega+\phi)+0+\sin^2(\omega+\phi)\right)$$

$$I_{22}(\theta) = \frac{1}{\sigma^2}\sum_{k=-1}^{1}\sin^2(\omega k+\phi) = \frac{1}{\sigma^2}\left(\sin^2(\omega+\phi)+\sin^2(\phi)+\sin^2(-\omega+\phi)\right)$$

$$\text{and } I(\theta) = \begin{bmatrix} I_{11}(\theta) & I_{12}(\theta) \\ I_{12}(\theta) & I_{22}(\theta) \end{bmatrix}$$

Can play some trig games here to get in terms of cosines, but this is sufficient.