ECE531 Spring 2013 Quiz 3

Your Name: SOLUTION

Instructions: This quiz is worth a total of 100 points. The quiz is open book and open notes. You may also use a calculator. You may not use a computer, phone, or tablet. Please show your work on each problem and box/circle your final answers. Points may be deducted for a disorderly presentation of your solution.

1. 40 points. Suppose you wish to estimate a two-coefficient multipath channel $h = [h_0, h_1]^\top \in \mathbb{R}^2$ by sending a known data sequence $\{x_0, x_1, \dots\}$ through the channel and observing the outputs of the channel. The observation at time k is given by

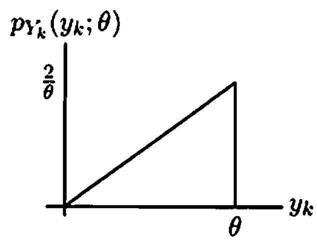
$$Y_k = h_0 x_k + h_1 x_{k-1} + W_k$$

with W_k corresponding to zero-mean Guassian noise. Given a known data sequence of $x_k = k$, a two-sample observation vector $Y = [Y_1, Y_2]^\top$, and a noise vector $W = [W_1, W_2]^\top$ with $W \sim \mathcal{N}(0, C)$ where

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 0.25 \end{bmatrix}$$

is the noise covariance, find an MVU estimator of the channel coefficients $h = [h_0, h_1]^\top$.

2. 60 points total. Suppose you receive observations $Y = [Y_0, \dots, Y_{n-1}]^\top$ with each individual observation drawn i.i.d. from the pdf shown below.



Note that the marginal pdf $p_{Y_k}(y_k;\theta) = \frac{2}{\theta^2} y_k I_{y\in[0,\theta]}$ where $I_{\mathcal{A}}$ is an indicator function equal to 1 when $\mathcal{A}$ is true and equal to zero otherwise.

(a) 20 points. Show that $T(y) = \max_k y_k$ is a sufficient statistic.

(b) 40 points. Assuming $T(y)$ is complete (you don't need to show it), use the RBLS theorem to find an MVU estimator of θ. You may find the following fact useful: letting $Z = \max_k Y_k$, we have

$$p_Z(z;\theta) = \frac{2nz^{2n-1}}{\theta^{2n}} I_{z\in[0,\theta]}$$

$\boxed{1.}$

$$Y_1 = h_0 x_1 + h_1 x_0 + W_1 = h_0 + W_1$$

$$Y_2 = h_0 x_2 + h_1 x_1 + W_2 = 2h_0 + h_1 + W_2$$

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} h_0 \\ h_1 \end{bmatrix}}_{\theta} + \underbrace{\begin{bmatrix} W_1 \\ W_2 \end{bmatrix}}_{W}$$

$$C^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0.25 \end{bmatrix} \Rightarrow D = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \qquad D^T D = C^{-1} \checkmark$$

so $\hat{\theta} = (H^T C^{-1} H)^{-1} H^T C^{-1} y$

just a matter of linear algebra now...

$$\hat{\theta}_{MVU} = \left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0.5 & 0.25 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0.5 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0.5 \\ 0.5 & 0.25 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0.5 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= 4 \begin{bmatrix} 0.25 & -0.5 \\ -0.5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0.5 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ -2 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0.5 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Interpretation: $\hat{h}_0 = y_1 \approx h_0 \checkmark$

$\hat{h}_1 = -2y_1 + y_2 \approx -2h_0 + 2h_0 + h_1 \approx h_1 \checkmark$

2.

a) write joint pdf $P_Y(y;\theta) = \prod_k \frac{2}{\theta^2} y_k I_{y \in [0,\theta]}$

$$= \left(\frac{2}{\theta^2}\right)^n \left(\prod_{k=0}^{n-1} y_k\right) I_{\min(y_k) \geq 0} \, I_{\max(y_k) \leq \theta}$$

$$= \underbrace{\left(\frac{2}{\theta^2}\right)^n I_{\max(y_k) \leq \theta}}_{g_\theta(T(y))} \underbrace{\left(\prod_{k=0}^{n-1} y_k\right) I_{\min(y_k) \geq 0}}_{h(y)}$$

$\Rightarrow T(y) = \max(y_k)$ is sufficient

b) Let $Z = \max Y_k$ (sufficient statistic)

$$E\{Z\} = \int_0^\theta z \cdot \frac{2n\, z^{2n-1}}{\theta^{2n}}\, dz$$

$$= \frac{2n}{\theta^{2n}} \int_0^\theta z^{2n}\, dz$$

$$= \frac{2n}{\theta^{2n}} \left[\frac{1}{2n+1} z^{2n+1}\right]_0^\theta$$

$$= \frac{2n}{2n+1} \frac{\theta^{2n+1}}{\theta^{2n}} = \frac{2n}{2n+1}\theta$$

so $\hat{\theta}(y) = \frac{2n+1}{2n} \max(y_k)$ is unbiased

Hence $E\{\hat{\theta}(Y) \mid T(Y) = T(y)\} = \frac{2n+1}{2n} \max(y_k) = \hat{\theta}_{MVU}(y)$

is the MVU estimator.