ECE531 Spring 2013 Quiz 4

Your Name: SOLUTION

Instructions: This quiz is worth a total of 100 points. The quiz is open book and open notes. You may also use a calculator. You may not use a computer, phone, or tablet. Please show your work on each problem and box/circle your final answers. Points may be deducted for a disorderly presentation of your solution.

For all of the questions below, consider the observation model

$$Y_k = \theta + W_k$$

for $k = 0, \ldots, n-1$ where the unknown parameter $\theta \in \mathbb{R}$ and where W_k are i.i.d. random variables drawn from

$$p_{W_k}(x) = \frac{1}{2}e^{-|x|}$$

for $-\infty < x < \infty$.

1. 30 points. For the case when $n = 2$, show that $\hat{\theta}_{\mathsf{ML},2}(y) = \frac{y_0+y_1}{2}$ is a maximum likelihood estimator. Is this maximum likelihood estimator unique? Explain.

   - Hint 1: The observation model in this problem is simple enough that you can just sketch $p_Y(y;\theta)$ as a function of θ.
   - Hint 2: Since the observations are i.i.d., their order doesn't matter. It may be helpful to assume $y_0 \le y_1$. This assumption does not reduce the generality of your answer.

2. 30 points. For the case when $n = 4$, is $\hat{\theta}_{\mathsf{ML},4}(y) = \frac{y_0+y_1+y_2+y_3}{4}$ a maximum likelihood estimator? Explain.

3. 40 points. Determine the asymptotic distribution of $\sqrt{n}(\hat{\theta}_{\mathsf{ML},n}(Y) - \theta)$ as $n \to \infty$. Be explicit.

   - Hint 3: For scalar Y, we have

   $$\frac{\partial}{\partial\theta}\ln p_Y(y\,;\,\theta) = \mathrm{sign}(y-\theta).$$

   Note that the derivative does not exist at $y = \theta$. Nevertheless, since $y = \theta$ with probability zero, the nonexistence of a derivative at this point does not affect expectations of $\frac{\partial}{\partial\theta}\ln p_Y(y\,;\,\theta)$ or functions thereof.

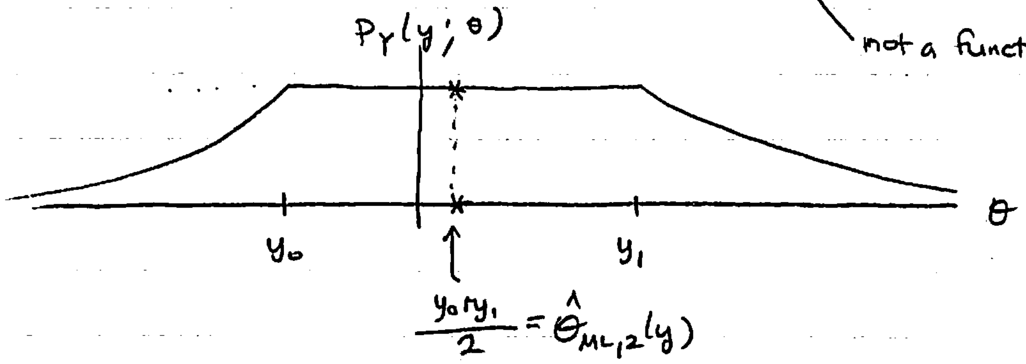
1. Given $n=2$, the joint pdf is

$$p_Y(y;\theta) = \frac{1}{4}\exp(-|y_0-\theta|)\exp(-|y_1-\theta|)$$

Since the order of the observations don't matter lets assume we've sorted the observations so that $y_0 \le y_1$. Then

$$p_Y(y;\theta) = \begin{cases} \frac{1}{4}\exp(-y_0-y_1+2\theta) & \theta < y_0 \\ \frac{1}{4}\exp(y_0-y_1) & y_0 \le \theta \le y_1 \\ \frac{1}{4}\exp(y_0+y_1-2\theta) & \theta > y_1 \end{cases}$$

not a function of θ here.



Clearly $\hat{\theta}_{ML,2}(y)$ is an MLE because it achieves a maximum of the likelihood function.

It is also clear that it is not unique unless $y_0 = y_1$.

2. Given $n=4$.

To check if this is an MLE, we need

$$h(y;\theta) = -|y_0 - \theta| - |y_1 - \theta| - |y_2 - \theta| - |y_3 - \theta|$$

to not be a function of θ.

Again, assuming $y_0 \le y_1 \le y_2 \le y_3$,

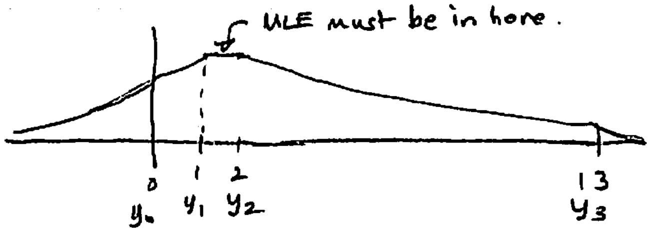
the MLE must produce an estimate $\in [y_1, y_2]$

Suppose $y_0 = 0$

$y_1 = 1$

$y_2 = 2$

$y_3 = 13$



Then $\dfrac{y_0 + y_1 + y_2 + y_3}{4} = 4 \notin [1, 2]$

So $\hat{\theta}_{ML,4}(y)$ is not an MLE.

3. Since iid observations, we need the Fisher information of one observation.

$$i(\theta) = E\left[\left(\frac{\partial \ln p_Y(y;\theta)}{\partial\theta}\right)^2\right] = E\left[\left(\frac{\partial}{\partial\theta}(-|y_0 - \theta|)\right)^2\right]$$

$$= E\left[(\pm 1)^2\right] = 1.$$

So

$$\sqrt{n}\left(\hat{\theta}_{ML,n} - \theta\right) \xrightarrow{d} N(0, 1).$$