ECE531 Spring 2013 Quiz 5

Your Name: SOLUTION

Instructions: This quiz is worth a total of 100 points. The quiz is open book and open notes. You may also use a calculator. You may not use a computer, phone, or tablet. Please show your work on each problem and box/circle your final answers. Points may be deducted for a disorderly presentation of your solution.

1. 70 points total. Suppose that an unknown random scalar parameter Θ is known to have a prior distribution of

$$p_\Theta(\theta) = \pi(\theta) = \begin{cases} e^{-\theta} & \theta > 0 \\ 0 & \theta \le 0. \end{cases}$$

You receive one scalar observation

$$Y = \Theta + W$$

with W independent of Θ and possessing the distribution

$$p_W(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x \le 0. \end{cases}$$

(a) 50 points. Find the posterior pdf of the scalar parameter given $Y = y$.

(b) 10 points. Using your answer to part (a), find the MMSE estimator of the unknown scalar parameter. Is the MMSE estimator unique?

(c) 10 points. Using your answer to part (a), find the MAP estimator of the unknown scalar parameter. Is the MAP estimator unique?

2. 30 points. Using the same observation model as Problem 1, find the ML estimator of the unknown scalar parameter and compare your answer to Problem 1. Discuss any differences. Hint: The ML estimator is appropriate for non-random parameter estimation, hence your analysis here should not use the prior on Θ.

1. a) First find the posterior pdf $\pi_y(\theta)$

$$\pi_y(\theta) = \frac{\pi(\theta)\, p_\theta(y)}{p(y)}$$

$\pi(\theta)$ given

$$p_\theta(y) = \begin{cases} e^{-(y-\theta)} & y > \theta \\ 0 & y \le \theta \end{cases}$$

To determine $p(y)$, we can use the total probability theorem (or convolve the pdfs)...

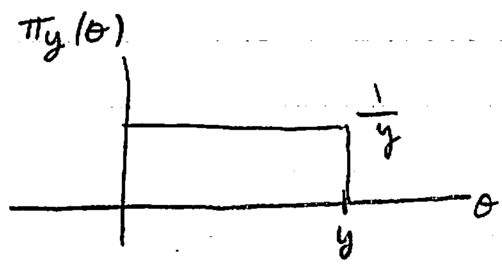
$$p_Y(y) = \int_{-\infty}^{\infty} p_\theta(y)\, \pi(\theta)\, d\theta$$

$$= \int_0^y e^{-(y-\theta)} e^{-\theta}\, d\theta$$

$$= \int_0^y e^{-y}\, d\theta = y e^{-y}$$

so $\pi_y(\theta) = \underset{\text{given observation}}{\underset{\text{of parameter}}{\text{posterior density}}} = \begin{cases} \dfrac{e^{-\theta} e^{-(y-\theta)}}{y e^{-y}} = \dfrac{1}{y} & 0 < \theta \le y \\ 0 & \text{otherwise.} \end{cases}$

so



uniformly distributed!

b) conditional mean is easy to compute now

$$\boxed{\hat{\theta}_{MMSE}(y) = E[\Theta \mid Y=y] = \frac{y}{2}}$$ unique

c) MAP estimator also easy

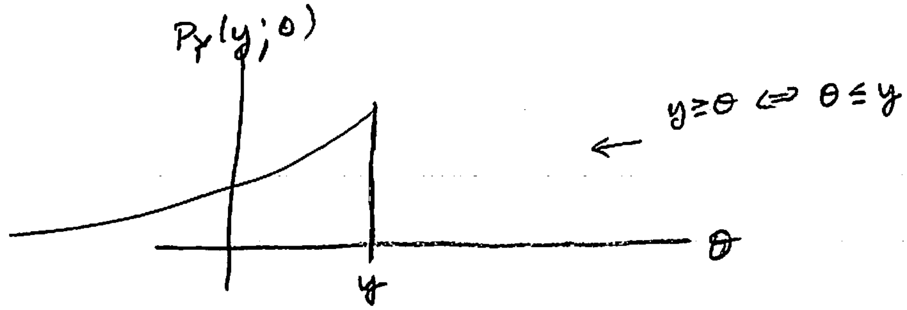
$$\boxed{\hat{\Theta}_{MAP}(y) \text{ is any value on } (0, y]}$$ not unique.

2. The MLE just θ that maximizes $p_Y(y;\theta)$

non-random

Here we have

$$p_Y(y;\theta) = \begin{cases} e^{-(y-\theta)} & y \geq \theta \\ 0 & \text{otherwise.} \end{cases}$$



hence $\hat{\theta}_{ML}(y) = y$ is the unique MLE.

The MLE estimates are twice as big as the MMSE estimates, which shows how the prior affects our estimates. The MLE is also equal to the largest MAP estimate, but MAP is not unique.