

ECE531 Spring 2013 Quiz 6

Your Name: SOLUTION

Instructions: This quiz is worth a total of 100 points. The quiz is open book and open notes. You may also use a calculator. You may not use a computer, phone, or tablet. Please show your work on each problem and box/circle your final answers. Points may be deducted for a disorderly presentation of your solution.

1. 60 points. Suppose that Θ is a scalar random parameter with prior distribution $\Theta \sim \mathcal{U}(-a, a)$ where $a > 0$ is known. Each observation Y_k is received as

$$Y_k = \Theta + W_k$$

with $W_k \stackrel{\text{i.i.d.}}{\sim} \mathcal{U}(-b, b)$ independent of Θ with $b > 0$ known. Given observations y_1, \dots, y_N , find the LMMSE estimator of Θ . Your answer should be in terms of a, b and the observations y_1, \dots, y_N . Discuss the cases $\frac{a}{b} \rightarrow 0$ and $\frac{a}{b} \rightarrow \infty$. Hint: You may find Woodbury's identity useful here.

2. 40 points. Consider a sequential LMMSE estimation scenario with scalar $\hat{\Theta}$ and observations $Y_k = h_k \Theta + W_k$ for $k = 0, 1, \dots$ with h_k known and $W_k \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$. Defining the error covariance

$$\Sigma[n] = \text{var} \left(\hat{\Theta}_{\text{LMMSE}}[n] \mid [Y_0, \dots, Y_n] = [y_0, \dots, y_n] \right) \in \mathbb{R}$$

Under what conditions does

- $\Sigma[n] > \Sigma[n - 1]?$
- $\Sigma[n] = \Sigma[n - 1]?$
- $\Sigma[n] < \Sigma[n - 1]?$

Explain your answer.

1. The LMMSE estimator is (from 12.6)

$$\hat{\theta}(y) = E[\Theta] + \text{Cor}_{\Theta Y}^{-1} (y - E[Y])$$

From the problem statement, we have $E[\Theta] = E[Y] = 0$.

We also have

$$\begin{aligned}\text{Cor}_{\Theta Y} &= E\{\Theta Y^T\} = E\{\Theta (\Theta 1 + W)^T\} = E\{\Theta^2\} 1^T + E\{\Theta W^T\} \\ &= \frac{4a^2}{12} 1^T + 0 = \frac{a^2}{3} 1^T\end{aligned}$$

and

$$C_{YY} = E\{YY^T\} = E\{(\Theta 1 + W)(\Theta 1 + W)^T\} = \frac{a^2}{3} 1 1^T + \frac{b^2}{3} I_N$$

To invert C_{YY} , we use Woodbury's identity

$$\begin{aligned}\left(\frac{b^2}{3} I_n + \left(\frac{a}{\sqrt{3}} 1\right) \left(\frac{a}{\sqrt{3}} 1\right)^T\right)^{-1} &= \frac{3}{b^2} I_n - \frac{\left(\frac{3}{b^2}\right)^2 \frac{a^2}{3}}{1 + \frac{a^2}{3} \frac{3}{b^2} 1^T 1} 1 1^T \\ &= \underbrace{\frac{3}{b^2} I_n - \frac{\left(\frac{3}{b^2}\right)^2 \left(\frac{a^2}{3}\right)}{1 + \left(\frac{a^2}{b^2}\right) N} 1 1^T}_{\text{Hence,}}\end{aligned}$$

Hence,

$$\begin{aligned}\hat{\theta}(y) &= 0 + \left(\frac{a^2}{3} 1^T\right) \cdot \left(\underbrace{\quad}_{\text{Let } \gamma = \frac{a^2}{b^2}}\right) (y - 0) \\ &= \left(\frac{a^2}{b^2} 1^T - \frac{\left(\frac{a^2}{b^2}\right)^2 N}{1 + \left(\frac{a^2}{b^2}\right) N} 1^T\right) y \\ &= \left(\gamma - \frac{\gamma^2 N}{1 + \gamma N}\right) 1^T y \\ &= \left(\frac{\gamma + \gamma^2 N}{1 + \gamma N} - \frac{\gamma^2 N}{1 + \gamma N}\right) 1^T y \\ &= \frac{\gamma}{1 + \gamma N} 1^T y = \boxed{\frac{\gamma N}{1 + \gamma N} \bar{y}} \quad \text{where } \bar{y} = \frac{1}{N} \sum_k y_k\end{aligned}$$

when $\gamma \rightarrow 0$, we have $\hat{\theta}(y) = 0$ (we ignore the observations because the prior is much better)

when $\gamma \rightarrow \infty$, we have $\hat{\theta}(y) = \bar{y}$ (we ignore the prior because the observations are much better).

$$2. \quad Y_K = h_K \oplus W_K \quad h_K \neq 0$$

Suppose $n \geq 1$

$$\text{we have } \Sigma[n] = (1 - K[n] h_n) \Sigma[n-1]$$

$$= \left(1 - \underbrace{\frac{\Sigma[n-1] h_n}{1 + h_n^2 \Sigma[n-1]} h_n}_{K[n] \text{ scalar}} \right) \Sigma[n-1]$$

$$\Rightarrow \Sigma[n] = \left(\frac{1 + h_n^2 \Sigma[n-1]}{1 + h_n^2 \Sigma[n-1]} - \frac{h_n^2 \Sigma[n-1]}{1 + h_n^2 \Sigma[n-1]} \right) \Sigma[n-1]$$

$$= \left(\frac{1}{1 + h_n^2 \Sigma[n-1]} \right) \Sigma[n-1]$$

If $h_n \neq 0$ (given) and $\Sigma[n-1] > 0$, then $\Sigma[n]$ must be strictly less than $\Sigma[n-1]$, since

$$\frac{1}{1 + \text{something positive}} < 1.$$

This should make intuitive sense since each new observation is giving more information, hence the estimator's error covariance must be strictly decreasing.

Only if $h_n = 0$ or $\Sigma[n-1] = 0$ do we get the case $\Sigma[n] = \Sigma[n-1]$.