ECE531 Spring 2013 Quiz 7

Your Name: SOLUTION

Instructions: This quiz is worth a total of 100 points. The quiz is open book and open notes. You may also use a calculator. You may not use a computer, phone, or tablet. Please show your work on each problem and box/circle your final answers. Points may be deducted for a disorderly presentation of your solution.

1. 100 points. Consider the *scalar* dynamical system

$$X[\ell+1] \quad = \quad fX[\ell] + U[\ell] \text{ for } \ell = 0, 1, \ldots$$

where f is a known scalar constant, and the observation model

$$Y[\ell] = hX[\ell] + V[\ell]$$

for $\ell = 0, 1, \ldots$ with $h = \frac{1}{\sqrt{2}}$. Assume that $\{U[0], U[1], \ldots\}$ and $\{V[0], V[1], \ldots\}$ are independent sequences of i.i.d. $\mathcal{N}(0,1)$ random variables. Also assume that the initial state $X[0] \sim \mathcal{N}(0, \sigma^2)$ and is independent of all $U[\ell]$ and $V[\ell]$.

(a) (50 points). Determine a value of the initial state variance $\sigma^2 > 0$ such that the Kalman gain is a constant for all $\ell = 0, 1, \ldots$, i.e.,

$$K[\ell] = \Sigma[\ell \mid \ell-1] H^\top[\ell] \left(H[\ell] \Sigma[\ell \mid \ell-1] H^\top[\ell] + R[\ell] \right)^{-1} \equiv K.$$

(b) (50 points). Find expressions for the prediction error variance (the mean squared prediction error) and the estimation error variance (the mean squared estimation error) for the case derived in Part (a). Comment on the behavior of these errors as $|f| \to 0$ and $|f| \to \infty$.

a) If K is a constant, then so is

$$\Sigma[\ell+1|\ell] = P = \Sigma[0|-1] = \sigma^2$$

$$\Sigma[\ell|\ell] = S$$

so $$K = \frac{\sigma^2 h}{\sigma^2 h^2 + 1} = \frac{\sqrt{2}\,\sigma^2}{\sigma^2+2} \qquad (0)$$

now $\Sigma[0|-1] = \Sigma[1|0] = \sigma^2$

$$\Sigma[1|0] = f^2\Sigma[0|0] + 1 \quad \Rightarrow \quad P = f^2 S + 1 = \sigma^2 \qquad (1)$$

$$S = \Sigma[0|0] = \Sigma[0|-1] - Kh\Sigma[0|-1]$$

$$= P - KhP = P\left(1-\frac{K}{\sqrt{2}}\right) = \sigma^2\left(1-\frac{K}{\sqrt{2}}\right) \qquad (2)$$

substitute (2) into (1) to get

$$\sigma^2 = f^2\sigma^2\left(1-\frac{K}{\sqrt{2}}\right) + 1$$

substitute (0) to get

$$\sigma^2 = f^2\sigma^2\left(1-\frac{\sigma^2}{\sigma^2+2}\right) + 1$$

solve for σ^2 ...

$$\sigma^2 = f^2\sigma^2\left(\frac{2}{\sigma^2+2}\right) + 1$$

$$\sigma^2 = \frac{2f^2\sigma^2}{\sigma^2+2} + 1$$

$$\sigma^4 + 2\underline{\sigma^2} - \underline{\sigma^2} - 2 - 2f^2\underline{\sigma^2} = 0$$

$$\sigma^4 - (2f^2-1) - 2 = 0$$

take unique positive root

$$\sigma^2 = \frac{2f^2-1 \pm \sqrt{(2f^2-1)^2+8}}{2} = \boxed{f^2-\frac{1}{2} + \sqrt{\left(f^2-\frac{1}{2}\right)^2+2}}$$

This value for the initial variance of $X[0]$ causes the Kalman filter to be in steady-state for all $\ell = 0, 1, \ldots$

b) The prediction error variance is simply $P = \sigma^2$

so $|f| \to 0 \Rightarrow \sigma^2 \to \sqrt{\frac{1}{4}+2} - \frac{1}{2} = 1$.

$|f| \to \infty \Rightarrow \sigma^2 \to 2f^2 \to \infty$

when $|f|=0$, the prediction error variance goes to 1 because we are just trying to predict the process noise.

when $|f| \to \infty$, the predictions become more difficult (the states are making large jumps) and the prediction error becomes unbounded.

The estimation error variance is S:

$$S = \sigma^2\left(1 - \frac{k}{\sqrt{2}}\right) = \sigma^2\left(1 - \frac{\sigma^2}{\sigma^2+2}\right) = \frac{2\sigma^2}{\sigma^2+2}$$

$|f| \to 0 \Rightarrow \sigma^2 \to 1 \Rightarrow S \to \frac{2}{3}$

$|f| \to \infty \Rightarrow \sigma^2 \to \infty \Rightarrow S \to 2$.

when $|f| \to 0$, the states are just the process noise, and there is no memory in the system. So this is effectively like estimating a unit variance Gaussian random variable

$$Y = hX + W \quad \text{all scalars}$$

$X \sim N(0,1)$, $W \sim N(0,1)$

$E[X|Y] = C_{XY} C_{YY}^{-1} y$ with $C_{XY} = h$ and $C_{YY} = h^2+1$

which has MSE $C_{XX} - C_{XY} C_{YY}^{-1} C_{YX} = 1 - \frac{h^2}{h^2+1} = \frac{1}{h^2+1}$

$$= \frac{1}{\frac{1}{2}+1} = \frac{2}{1+2} = \frac{2}{3} \checkmark$$

When $|f| \to \infty$, the predictions become useless, but the estimates still have finite variance. In fact, since $k \to \frac{1}{h}$

$\hat{X}[\ell|\ell] = \frac{1}{h} Y[\ell] = X[\ell] + \sqrt{2}V[\ell]$

we see that the variance of the estimate is just the variance of the measurement noise.