ECE531 Spring 2013 Quiz 9

Your Name: SOLUTION

Instructions: This quiz is worth a total of 100 points. The quiz is open book and open notes. You may also use a calculator. You may not use a computer, phone, or tablet. Please show your work on each problem and box/circle your final answers. Points may be deducted for a disorderly presentation of your solution.

Suppose you receive an observation Y drawn either from conditional distribution $p_0(y)$ or conditional distribution $p_1(y)$ shown in Figure 1. Hypothesis $\mathcal{H}_i$ is that the observation is drawn from $p_i(y)$ for $i \in \{0, 1\}$.

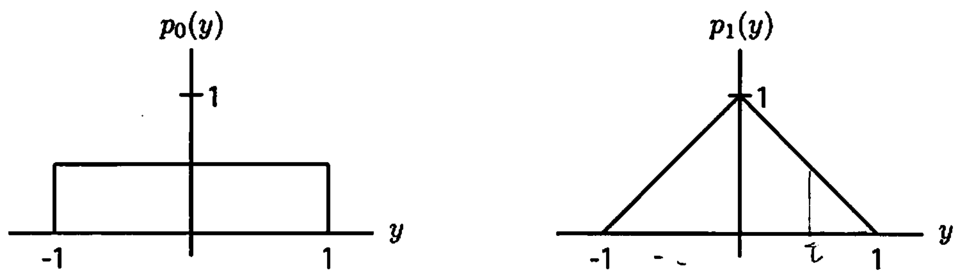
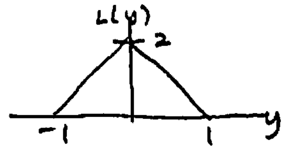


Figure 1: Conditional distributions for Problem 2.

1. 40 points. Find the Neyman-Pearson detector with false positive probability constraint $P_{\mathsf{fp}} \leq \alpha$ for general $0 \leq \alpha \leq 1$.
2. 10 points. Compute the probability of detection for your detector from part (a) as a function of $0 \leq \alpha \leq 1$ and sketch the ROC.
3. 40 points. Assume the uniform cost assignment and find a Bayes decision rule for a general prior $\{\pi_0, \pi_1\}$ with $\pi_1 = 1 - \pi_0$ and $0 \leq \pi_0 \leq 1$.
4. 10 points. Compute the probability of error of your detector from part (d) as a function of $0 \leq \pi_0 \leq 1$.

In all cases, please simplify and describe your decision rules and make them as explicit as you can.

1. $L(y) = \frac{p_1(y)}{p_0(y)} = 2p_1(y).$



so our decision rule will be of the form

$$\rho^{NP}(y) = \begin{cases} 1 & \text{if } |y| < \tau \\ 0 & \text{otherwise.} \end{cases}$$

The threshold τ can be computed as

$$P_{fp} = \int_{-\tau}^{\tau} p_0(y)\,dy = \tau = \alpha, \qquad \text{So } \tau = \alpha$$

Hence

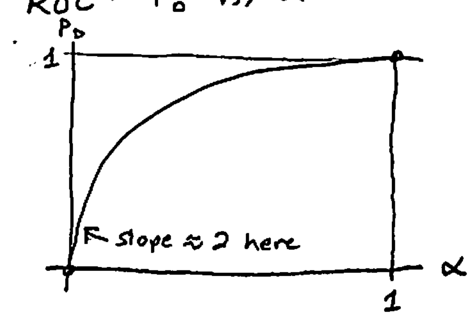
$$\boxed{\rho^{NP}(y) = \begin{cases} 1 & \text{if } |y| < \alpha \\ 0 & \text{otherwise.} \end{cases}}$$

2. $P_D = \int_{-\tau}^{\tau} p_1(y)\,dy = 2\int_0^{\tau} p_1(y)\,dy = 2\int_0^{\tau} 1-y\,dy = 2\left[y - \frac{y^2}{2}\right]_0^{\tau} = 2\left(\tau - \frac{\tau^2}{2}\right)$

but $\tau = \alpha$, so

$$\boxed{P_D = 2\left(\alpha - \frac{\alpha^2}{2}\right)}$$

ROC: P_D vs. α



3. UCA so

$$\delta^{B\pi} = \begin{cases} 1 & L(y) > \frac{\pi_0}{1-\pi_0} \\ 1/0 & L(y) = \,.. \\ 0 & L(y) < \frac{\pi_0}{1-\pi_0} \end{cases}$$

but $L(y) = \begin{cases} 2 - 2|y| & y \in [-1, 1] \\ \text{undefined} & \text{otherwise} \end{cases}$

$$2 - 2|y| > \frac{\pi_0}{1-\pi_0} \iff 2 - \frac{\pi_0}{1-\pi_0} > 2|y| \iff |y| < 1 - \frac{\pi_0}{2(1-\pi_0)} = \tau$$

so the Bayes decision rule is

$$\boxed{\delta^{B\pi} = \begin{cases} 1 & |y| \le 1 - \frac{\pi_0}{2(1-\pi_0)} \\ 0 & \text{otherwise.} \end{cases}}$$

Same form as N-P
just a different threshold.
$\Rightarrow$ Note for $\pi_0 > 2/3$,
we have a negative
threshold $\Rightarrow$ always decide H_0

4. $P_e = \text{Prob}(\text{decide } H_1 \mid x_0)\,\pi_0 + \text{Prob}(\text{decide } H_0 \mid x_1)\,\pi_1$

$= \frac{1}{2}(2\tau)\pi_0 + 2\cdot\frac{1}{2}(1-\tau)^2(1-\pi_0) \leftarrow$ (Assumes $\pi_0 \le 2/3$)

$= \left(1 - \frac{\pi_0}{2(1-\pi_0)}\right)\pi_0 + \left(\frac{\pi_0}{2(1-\pi_0)}\right)^2(1-\pi_0) = \pi_0 - \frac{\pi_0^2}{2(1-\pi_0)} + \frac{\pi_0^2}{4(1-\pi_0)} = \boxed{\pi_0 - \frac{\pi_0^2}{4(1-\pi_0)}}$

check: $\pi_0 = 0 \Rightarrow P_e = 0$, $\pi_0 = \frac{2}{3} \Rightarrow P_e = \frac{2}{3} - \frac{4/9}{4(1/3)} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} = 1 - \pi_0$

for $\pi_0 > 2/3$, we always decide H_0, so $P_e = 1 - \pi_0$