SINR, Power Efficiency, and Theoretical System Capacity of Parallel Interference Cancellation

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Abstract— **In this paper, we analytically derive an exact expression for the SINR of the two-stage parallel interference cancellation (PIC) multiuser detector in a synchronous, nonorthogonal, binary, CDMA communication system. In order to obtain an intuitive understanding, we consider an approximation to the SINR expression that is well justified in scenarios where the error probability of the matched filter detector is reasonably low. Given a specific SINR requirement for each user in the system, we derive an expression for the minimum transmit power necessary to meet this requirement when the two-stage PIC detector is used. We also derive an expression for a measure of the theoretical system capacity of PIC, defined as the maximum number of users possible in a system with finite available transmit power. Analytical results comparing PIC to the successive interference cancellation (SIC) detector and matched filter (MF) detector show that PIC requires less total transmit power and has greater theoretical system capacity than the SIC or MF detectors in the cases considered.**

I. Introduction

One promising technique for mitigating multiple access interference in CDMA communication systems is parallel interference cancellation (PIC). PIC was first introduced for CDMA communication systems in [1] and [2] as the multistage detector and was shown to have close connections to joint maximum likelihood detection. Since Varanasi and Aazhang's pioneering work, there has been an increased interest in understanding the performance of the PIC detector (see, for instance, [3], [4], [5], [6], [7]).

In this paper, we study the signal to interference plus noise ratio (SINR) performance of PIC and its implications to transmit power and theoretical system capacity. We present an exact expression for the SINR of the PIC detector and, in order to obtain a more intuitive understanding, we suggest an approximation that holds in typical operating scenarios where the error probability of the matched filter detector is reasonably low. Using this result, we consider the case where each user in the CDMA communication system has a particular SINR requirement and derive an expression for the minimum transmit powers necessary to satisfy these requirements. Note that, in a nonorthogonal multiuser system such as CDMA, increasing one user's transmit power to meet their SINR requirement can also have the effect of increasing the interference seen by the other users in the system, hence lowering their SINR. The approximations used in this paper allow the transmit powers to be computed via a set of of simultaneous linear equations.

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As a first step towards understanding the SINR performance of PIC, we derive closed form expressions for the total required transmit power and theoretical system capacity of the PIC detector in the equi-correlated case where all users have identical signature waveform crosscorrelations. These expressions are analytically compared to the results for SIC and MF detection provided in [8] under identical assumptions. We provide analytical proofs that show that, in the cases considered, PIC requires less total transmit power and has greater theoretical system capacity than the SIC or MF detectors. Numerical results verifying the analysis are also presented.

We assume a synchronous CDMA multiuser communication scenario with binary signaling, nonorthogonal transmissions, and an additive white Gaussian noise channel. The communication system model is identical to the basic synchronous CDMA model described in [9]. The number of users in the system is denoted by K and all receivers considered in this paper operate on the K-dimensional MF bank output given by the expression

$$\boldsymbol{y} = \boldsymbol{R}\boldsymbol{A}\boldsymbol{b} + \sigma\boldsymbol{n} \tag{1}$$

where $\boldsymbol{R} \in \mathbb{R}^{K\times K}$ is a symmetric matrix of normalized user signature crosscorrelations such that $\boldsymbol{R}_{kk} = 1$ for $m = 1, \ldots, K$ and $|\boldsymbol{R}_{k\ell}| \leq 1$ for all $k \neq \ell$, $\boldsymbol{A} \in \mathbb{R}^{K\times K}$ is a diagonal matrix of positive real amplitudes, $\boldsymbol{b} \in \mathbb{B}^{K\times 1}$ is the vector of i.i.d. equiprobable binary user symbols where $\mathbb{B} = \{\pm 1\}$, σ is the standard deviation of the additive channel noise, and $\boldsymbol{n} \in \mathbb{R}^{K\times 1}$ represents a matched filtered, unit variance AWGN process where $\mathrm{E}[\boldsymbol{n}] = \boldsymbol{0}$ and $\mathrm{E}[\boldsymbol{n}\boldsymbol{n}^\top] = \boldsymbol{R}$. The channel noise and user symbols are assumed to be independent.

II. SINR of Two-Stage PIC

Under the common assumption that the PIC detector has perfect knowledge of the amplitudes and signature crosscorrelations, the PIC detector forms the soft output for the k^{th} user by the expression

$$y_{\mathsf{PIC}}^{(k)} = a^{(k)}b^{(k)} + \sum_{\ell \neq k} \rho_{k\ell} a^{(\ell)} \underbrace{[b^{(\ell)} - \mathrm{sgn}(y^{(\ell)})]}_{\epsilon \in \{-2,0,2\}} + \sigma n^{(k)}.$$

The notation $\mathrm{SINR}_{\mathsf{X}}^{(k)}$ denotes the SINR of the k^{th} user's output of multiuser detector X defined as

$$\mathrm{SINR}_{\mathsf{X}}^{(k)} \triangleq \frac{\mathrm{E}[y_{\mathsf{X}}^{(k)} \mid b^{(k)}]^2}{\mathrm{var}[y_{\mathsf{X}}^{(k)} \mid b^{(k)}]}$$

where $y_{\mathsf{X}}^{(k)}$ denotes the k^{th} user's soft output from multiuser detector X prior to hard decision. We can then compute the SINR of the PIC detector as

$$\mathrm{SINR}_{\mathsf{PIC}}^{(k)} = \frac{\left(a^{(k)}b^{(k)} + \sum_{\ell \neq k} \rho_{k\ell} a^{(\ell)} \Psi_\ell\right)^2}{\sum_{\ell \neq k} \sum_{m \neq k} \rho_{k\ell}\rho_{km} a^{(\ell)} a^{(m)} \Omega_{\ell m} + 2\sigma \sum_{\ell \neq k} \rho_{k\ell} a^{(\ell)} \Phi_{\ell k} + \sigma^2}$$

where

$$\begin{aligned} \Psi_\ell &= \mathrm{E}[\epsilon^{(\ell)} \mid b^{(k)}], \\ \Omega_{\ell m} &= \mathrm{E}[\epsilon^{(\ell)}\epsilon^{(m)} \mid b^{(k)}] - \mathrm{E}[\epsilon^{(\ell)} \mid b^{(k)}]\mathrm{E}[\epsilon^{(m)} \mid b^{(k)}], \text{ and} \\ \Phi_{\ell k} &= \mathrm{E}[\epsilon^{(\ell)} n^{(k)} \mid b^{(k)}]. \end{aligned}$$

Exact expressions for Ψ, Ω, and Φ are given in the Appendix of this paper. It is evident that the exact expression for the SINR of the PIC detector is quite complex and does not lead to an intuitive understanding of its properties. Instead, we will impose the following "normal-operating" assumptions also imposed in [8] and indirectly in [9, pp. 378]:

1. Assume that $\epsilon^{(\ell)}$ is approximately independent of $b^{(k)}$ for all $\ell \neq k$, or in other words that an error in the decision of the matched filter output for user ℓ is independent of the bit transmitted by user k.
2. Assume that $\epsilon^{(\ell)}$ is approximately independent of $\epsilon^{(m)}$ for all $\ell \neq m$, or in other words that matched filter decision errors for user ℓ are independent of matched filter decision errors for user m.
3. Assume that $\epsilon^{(\ell)}$ is approximately independent of $n^{(k)}$ for all $\ell \neq k$, or in other words that matched filter decision errors for user ℓ are independent of the Gaussian channel noise in the k^{th} user's soft matched filter output.

These assumptions are well justified unless the error probabilities at the output of the matched filter detector are high and they imply that

$$\begin{aligned} \Psi_\ell &\approx 0 \quad \forall \ell \neq k \\ \Omega_{\ell m} &\approx 0 \quad \forall (\ell \neq k) \neq (m \neq k) \\ \Phi_{\ell k} &\approx 0 \quad \forall \ell \neq k. \end{aligned}$$

The remaining term requiring calculation is $\Omega_{\ell\ell}$ which can be derived as

$$\begin{aligned} \Omega_{\ell\ell} &= \mathrm{E}[(\epsilon^{(\ell)})^2 \mid b^{(k)}] - \mathrm{E}[\epsilon^{(\ell)} \mid b^{(k)}]^2 \\ &\approx \mathrm{E}[(\epsilon^{(\ell)})^2] - 0 = 4P_e^{(\ell)} \end{aligned}$$

where $P_e^{(\ell)} = P(b^{(\ell)} \neq \mathrm{sgn}(y^{(\ell)}))$ is the probability of error of the ℓ^{th} user's matched filter output and where the imposed assumptions were used in the approximation. Under these approximations, the SINR of the PIC detector may then be written as

$$\mathrm{SINR}_{\mathsf{PIC}}^{(k)} = \frac{\alpha^{(k)}}{\sum_{\ell \neq k} r_{k\ell}\alpha^{(\ell)} 4P_e^{(\ell)} + 1}. \tag{2}$$

where $\alpha^{(k)} = (a^{(k)}/\sigma)^2$ is the normalized power (or SNR) of the k^{th} user and $r_{k\ell} = \rho_{k\ell}^2$ is the squared crosscorrelation of the k^{th} and ℓ^{th} users' signature waveforms.

III. Power Efficiency

In this section, we use the result of (2) to calculate the normalized power required by each user in the system in order to meet a particular SINR requirement. If we define the normalized power vector $\boldsymbol{\alpha} = [\alpha^{(1)}, \ldots, \alpha^{(K)}]^\top$ then (2) implies that

$$\begin{aligned} \boldsymbol{\alpha} &= [4\boldsymbol{S}(\boldsymbol{\Gamma} - \boldsymbol{I})\boldsymbol{P}]\boldsymbol{\alpha} + \boldsymbol{S}\boldsymbol{e} \\ &= [\boldsymbol{I} - 4\boldsymbol{S}(\boldsymbol{\Gamma} - \boldsymbol{I})\boldsymbol{P}]^{-1}\boldsymbol{S}\boldsymbol{e} \end{aligned} \tag{3}$$

where $\boldsymbol{S}$ is a diagonal matrix with the $\ell\ell^{th}$ element equal to the output SINR requirement for user ℓ, $\boldsymbol{P}$ is a diagonal matrix with the $\ell\ell^{th}$ element equal to $P_e^{(\ell)}$, $\boldsymbol{\Gamma}$ is a matrix of squared signature crosscorrelations given as

$$\boldsymbol{\Gamma} = \begin{bmatrix} r_{11} & \cdots & r_{1K} \\ \vdots & \ddots & \vdots \\ r_{K1} & \cdots & r_{KK} \end{bmatrix}$$

where $r_{\ell\ell} = 1$ for all ℓ, and $\boldsymbol{e}$ is a K-vector with all elements equal to one. If the inverse exists in (3) then there is a unique solution for the normalized user powers given $\boldsymbol{S}$, $\boldsymbol{\Gamma}$, and $\boldsymbol{P}$. Also note that when the interference cancellation is perfect (i.e., when $\boldsymbol{P} = \boldsymbol{0}$) then $\boldsymbol{\alpha} = \boldsymbol{S}\boldsymbol{e}$ or, in other words, each user's normalized power (SNR) is equal to their output SINR requirement. This is intuitively satisfying since perfect cancellation implies that there is no MAI in the outputs of the two-stage PIC detector and that noise is the only channel impairment.

Proposition 1: Under the following assumptions:

1. The squared user crosscorrelations are all identical, i.e., $r_{k\ell} = r$ for all $k \neq \ell$,
2. The user output SINR requirements are all identical, i.e., $\boldsymbol{S} = s\boldsymbol{I}$, and
3. The decision error probabilities are all identical, i.e., $\boldsymbol{P} = p\boldsymbol{I}$,

then the total transmit power required for two-stage PIC detection may be written as

$$\boldsymbol{e}^\top \boldsymbol{\alpha} = \frac{Ks}{1 - 4rsp(K-1)} \tag{4}$$

Proof: Under the assumptions of the proposition, we can rewrite (3) as

$$\boldsymbol{\alpha} = s\underbrace{[\boldsymbol{I} - 4sp\boldsymbol{\Lambda}]}_{\boldsymbol{\Delta}}^{-1}\boldsymbol{e} \qquad (5)$$

where $\boldsymbol{\Lambda}$ is defined such that its diagonal elements are all equal to zero and its off-diagonal elements are all equal to r. The inverse in this last expression can be computed explicitly since $\boldsymbol{\Delta}$ has explicit solutions to its eigenvalues and eigenvectors. Denoting $x = -4rsp$, it can be shown that the diagonal elements of $\boldsymbol{\Delta}^{-1}$ are all identical and equal to $\frac{x(2-K)-1}{x^2(K-1)+x(2-K)-1}$ and that the off-diagonal elements of $\boldsymbol{\Delta}^{-1}$ are all identical and equal to $\frac{x}{x^2(K-1)+x(2-K)-1}$. It then follows directly that

$$\alpha^{(k)} = \frac{s(x(2-K)-1+(K-1)x)}{x^2(K-1)+x(2-K)-1}.$$

Recognizing that the numerator and denominator have the common factor $x-1$, we can simplify this last expression to write

$$\alpha^{(k)} = \frac{s}{(K-1)x+1}$$

hence the total normalized power required for the PIC detector is given in (4) after the substitution $x = -4rsp$. ■

The following remarks expose some of the intuitive properties of (4):

- The cases of perfect cancellation ($p = 0$) or orthogonal transmission ($r = 0$) are identical and lead to a total normalized power requirement of Ks.
- For fixed K, nonzero crosscorrelation or nonzero error probabilities lead to a penalty term in the denominator of (4) that leads to an increase in the total power required.

IV. System Capacity

A theoretical measure of system capacity, denoted as K_{max}, can be defined as the operating point at which the required power is infinite, or equivalently, when the denominator of (4) equals zero. In this case, we can state that, for the PIC detector,

$$K_{max} = \frac{1}{4rsp} + 1 \qquad (6)$$

which implies that the system capacity is approximately inversely proportional to the squared signature crosscorrelations r, the required output SINR s, and the error probability p of the MF first stage.

V. Comparison to MF and SIC Multiuser Detectors

Using the two-stage PIC SINR results derived in the prior section and the SINR results on SIC and MF multiuser detectors from [8], we can form Table I to compare the total power required ($\boldsymbol{e}^\top\boldsymbol{\alpha}$) and system capacity ($K_{max}$) of the PIC, SIC, and MF detectors for a given SINR requirement under the assumptions of Proposition 1. We note that the expressions for total power and

Detector	$\boldsymbol{e}^\top\boldsymbol{\alpha}$	K_{max}
MF	$\frac{Ks}{1-rs(K-1)}$	$\frac{1}{rs}+1$
SIC	$\frac{\theta^K-1}{r(1-4p\theta^K)}$	$\frac{-\log(4p)}{\log\theta}$
PIC	$\frac{Ks}{1-4rsp(K-1)}$	$\frac{1}{4rsp}+1$

TABLE I

Multiuser detector comparison under the assumptions of Proposition 1. $\theta \triangleq \frac{1+rs}{1+4rsp}$.

system capacity for the SIC detector are simplified but equivalent to the expressions presented in [8].

Comparison of the PIC and MF detectors is straightforward. A system using PIC detection requires less total transmit power $\boldsymbol{e}^\top\boldsymbol{\alpha}$ and has a higher K_{max} than a system with MF detection when $p < 0.25$ for any admissible values of K, r, and s. Conversely, for $p > 0.25$, a system using MF detection requires lower total transmit power and has a higher K_{max} than PIC for any K, r, and s. Since an error probability $p > 0.25$ describes an unusual operating region where communication has very low reliability, we can say roughly that the PIC detector is uniformly superior to the MF detector in terms of SINR, total required power, and system capacity in the equi-correlated case.

We compare PIC and SIC detectors in the following propositions.

Proposition 2: Under the same assumptions as Proposition 1 and $K \geq 2$, PIC requires less total transmit power when $0 \leq p < 0.25$ and $rs > 0$.

Proof: To show that PIC requires less total power than SIC we will show that

$$\frac{Ks}{1-4rsp(K-1)} < \frac{\theta^K - 1}{r(1-4p\theta^K)}$$

for $\theta \triangleq \frac{1+rs}{1+4rsp}$. For notational convenience we define

$$\begin{aligned} q &= 4p \\ \lambda &= rs \end{aligned}$$

and we also assume that all parameters are such that both denominators are positive in order for this comparison to make any sense. In this case we can cross multiply the expressions to get the following equivalent expression

$$K\lambda(1-q\theta^K) \quad < \quad (\theta^K - 1)(1 - q\lambda(K-1))$$

for $\theta = \frac{1+\lambda}{1+q\lambda}$ and collection of like terms yields

$$K\lambda(1-q) + 1 + q\lambda \quad < \quad \theta^K(1+q\lambda). \tag{7}$$

It can be shown that (7) holds for $K = 2$ under the assumptions of the proposition. To show that (7) also holds for arbitrary K we will use an inductive proof. Assume that (7) holds for some value of $K-1$. Then we will show that it also holds for K. The hypothesis of the induction implies that

$$(K-1)\lambda(1-q) + 1 + q\lambda \quad < \quad \theta^{K-1}(1+q\lambda)$$

for a particular value of $K-1$. Multiplying both sides by θ, the hypothesis implies that

$$\theta[(K-1)\lambda(1-q) + 1 + q\lambda] \quad < \quad \theta^K(1+q\lambda).$$

This last expression, combined with (7), implies that it is sufficient to show that

$$K\lambda(1-q) + 1 + q\lambda \quad < \quad \theta[(K-1)\lambda(1-q) + 1 + q\lambda]$$

in order to prove the claim. Using the fact that $\theta(1 + q\lambda) = 1 + \lambda$ we can write an equivalent expression

$$K\lambda(1-q) + 1 + q\lambda - 1 - \lambda \quad < \quad \theta(K-1)\lambda(1-q)$$

and simplifying

$$K\lambda(1-q) - \lambda(1-q) \quad < \quad \theta(K-1)\lambda(1-q)$$

which leads to the common positive factor $\lambda(1-q)$ hence

$$(K-1) \quad < \quad \theta(K-1)$$

which holds for $\theta > 1$ and $K \geq 2$. But $\theta > 1$ is equivalent to $0 \leq q < 1$ or, equivalently, $0 \leq p < 0.25$ hence (7) is true for K under the hypothesis that it is true for $K-1$. Since (7) can be shown explicitly true for $K = 2$ the claim is proven inductively. ∎

Proposition 3: Under the same assumptions as Proposition 1 and $K \geq 2$, PIC and has greater K_{max} than SIC when $0 < p < 0.25$ and $rs > 0$.

Proof: To prove this proposition, we first note that for $p = 0$ the denominator of the total power expressions for both the PIC and SIC detectors can never go to zero as K increases, hence both algorithms have theoretically infinite capacity when the decision error probability is zero. In order to prove that K_{max} is greater for PIC than SIC when $0 < p < 0.25$, we wish to show under our previously established notation that

$$\frac{1}{q\lambda} + 1 > \frac{-\log q}{\log \theta}.$$

Since $q\lambda > 0$ and $\log \theta > 0$ we can express this inequality equivalently as

$$q\lambda \log q > (1 + q\lambda) \log \frac{1 + q\lambda}{1 + \lambda}$$

where we have substituted $\theta = \frac{1+\lambda}{1+q\lambda}$. Defining

$$h(\lambda, q) = q\lambda \log q - (1 + q\lambda) \log \frac{1 + q\lambda}{1 + \lambda}$$

then it is equivalent to prove that $h(\lambda, q) > 0$ for all $\lambda > 0$ and $0 < q < 1$. To show this, we note that

$$\lim_{q \downarrow 0} h(\lambda, q) = 0 - \log \frac{1}{1 + \lambda} = \log(1 + \lambda) > 0 \tag{8}$$

and that

$$\lim_{q \uparrow 1} h(\lambda, q) \quad = \quad 0. \tag{9}$$

Since $h(q, \lambda)$ is continuous in q on the open interval $0 < q < 1$ for $\lambda > 0$, we can compute its partial derivative in this region as

$$\frac{\partial}{\partial q} h(\lambda, q) \quad = \quad \lambda \log \frac{q + q\lambda}{1 + q\lambda} < 0$$

where the inequality follows directly from the the assumptions $0 < q < 1$ and $\lambda > 0$. This implies that $h(\lambda, q)$ is monotonically decreasing on the open interval $0 < q < 1$ and this fact combined with (8) and (9) implies that $h(\lambda, q) > 0$ on the open interval $0 < q < 1$. ∎

Intuitively, PIC detection tends to outperform both SIC and MF detection in the $0 \leq p < 0.25$ interval because the decisions from the first stage are reliable enough such that cancellation is beneficial to the final decision statistics at the output of the second stage. The first user in a SIC detector is actually decided via MF detection and their decision statistic is subject to the interference of all of the other users. The second user's decision statistic is subject to $K-1$ interference terms and so forth. PIC detection attempts to cancel all of the multiple access interference for each user hence, when the interference estimates are reliable ($0 \leq p < 0.25$), the results of this analysis imply that, under operating conditions that satisfy the approximations used to derive (2) and the assumptions of Proposition 1, better performance can be achieved by canceling all of the multiple access interference in parallel rather than successively.

VI. Numerical Results

Although the prior section analytically showed that PIC detection requires less total transmit power and provides greater theoretical system capacity than SIC and MF detection, this section presents numerical examples that demonstrate that the actual performance difference may be quite significant. Figure 1 plots the total normalized power from Table I for PIC, SIC, and MF detectors as a function of K for several values of p. Note that the MF power requirements do not change as a function of decision error probability since there is no interference cancellation. These results clearly show that PIC detection may require several orders of magnitude less power than the SIC and MF detectors in the cases considered.

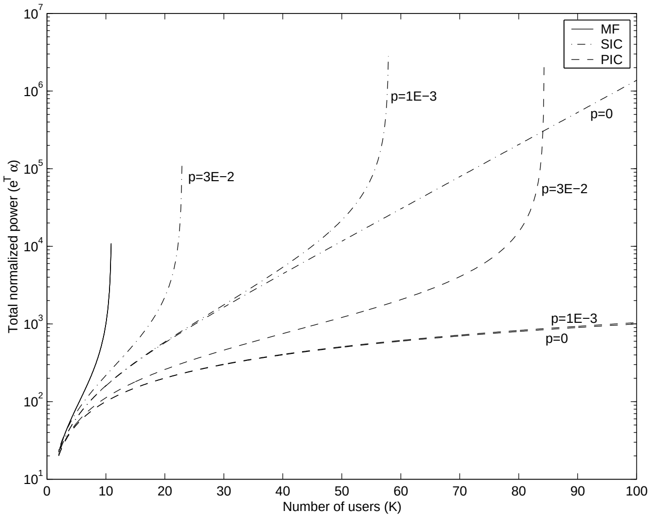

Fig. 1. Normalized total power required for PIC, SIC, and MF detection to meet the SINR requirement $s = 10$ for $r = 0.01$.

Figure 2 plots the theoretical system capacity expressions from Table I for PIC, SIC, and MF detectors as a function of p for two values of s. These results clearly show that PIC may offer several orders of magnitude greater theoretical system capacity than the SIC and MF detectors in the cases considered.

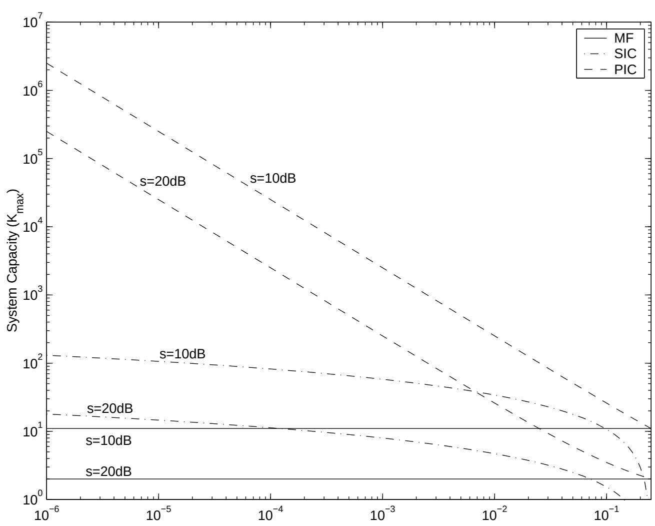

Fig. 2. System capacity for PIC, SIC, and MF detection to meet the SINR requirements $s = 10$dB and $s = 20$dB for $r = 0.01$.

VII. Conclusions

In this paper we derived an expression for the SINR of the PIC detector and examined its implications on power efficiency and theoretical system capacity. In the case where all users have the same SINR requirement and where the signature correlations are identical between all users, we showed analytically that PIC outperforms SIC and MF detection in terms of power efficiency and theoretical system capacity. Numerical results suggest that the performance differences may be significant.

Appendix

In this Appendix, we present the exact expressions for the terms used to calculate the SINR of the PIC detector.

I. Ψ_ℓ for $\ell \neq k$

Recall that $\Psi_\ell = \mathrm{E}[b^{(\ell)} - \mathrm{sgn}(y^{(\ell)}) \,|\, b^{(k)}]$. Since the users' bits are assumed independent and zero mean then $\Psi_\ell = -\mathrm{E}[\mathrm{sgn}(y^{(\ell)}) \,|\, b^{(k)}]$. Conditioning temporarily on all of the users' bits, we can write

$$\begin{aligned}\mathrm{E}[\mathrm{sgn}(y^{(\ell)}) \,|\, \boldsymbol{b}] &= P(y^{(\ell)} > 0 \,|\, \boldsymbol{b}) - P(y^{(\ell)} < 0 \,|\, \boldsymbol{b}) \\ &= Q\left(\frac{-\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}\right) - \left(1 - Q\left(\frac{-\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}\right)\right) \\ &= 1 - 2Q\left(\frac{\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}\right)\end{aligned}$$

where we have used the facts that $y^{(\ell)} = \boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b} + \sigma n^{(\ell)}$ and $Q(x) + Q(-x) = 1$. To remove the conditioning on $\boldsymbol{b}$, first denote $\mathcal{B}^{(k)}$ as the set of cardinality 2^{K-1} of all possible, equiprobable, binary K-vectors with the k^{th} user's bit fixed to the known value $b^{(k)}$. Then it follows that

$$\begin{aligned}\mathrm{E}[\mathrm{sgn}(y^{(\ell)}) \,|\, b^{(k)}] &= \frac{1}{2^{K-1}} \sum_{\boldsymbol{b} \in \mathcal{B}^{(k)}} \left(1 - 2Q\left(\frac{\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}\right)\right) \\ &= 1 - \frac{1}{2^{K-2}} \sum_{\boldsymbol{b} \in \mathcal{B}^{(k)}} Q\left(\frac{\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}\right)\end{aligned}$$

and Ψ_ℓ follows directly.

II. $\Omega_{\ell m}$ for $(\ell \neq k) \neq (m \neq k)$

Recall that

$$\begin{aligned}\Omega_{\ell m} &= \mathrm{E}[(b^{(\ell)} - \mathrm{sgn}(y^{(\ell)}))(b^{(m)} - \mathrm{sgn}(y^{(m)})) \,|\, b^{(k)}] - \Psi_\ell \Psi_m \\ &= \mathrm{E}[b^{(\ell)} b^{(m)} \,|\, b^{(k)}] - \mathrm{E}[b^{(\ell)} \mathrm{sgn}(y^{(m)}) \,|\, b^{(k)}] \\ &\quad - \mathrm{E}[b^{(m)} \mathrm{sgn}(y^{(\ell)}) \,|\, b^{(k)}] + \mathrm{E}[\mathrm{sgn}(y^{(\ell)}) \mathrm{sgn}(y^{(m)}) \,|\, b^{(k)}] \\ &\quad - \Psi_\ell \Psi_m.\end{aligned}$$

Since the users' bits are assumed independent and zero mean, $\mathrm{E}[b^{(\ell)} b^{(m)} \,|\, b^{(k)}] = 0$. To compute $\mathrm{E}[b^{(\ell)} \mathrm{sgn}(y^{(m)}) \,|\, b^{(k)}]$, we can temporarily condition on $\boldsymbol{b}$ to use a prior result in this Appendix to write

$$\mathrm{E}[b^{(\ell)} \mathrm{sgn}(y^{(m)}) \,|\, \boldsymbol{b}] = b^{(\ell)} \left[1 - 2Q\left(\frac{\boldsymbol{r}_m^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}\right)\right].$$

Now, removing the conditioning on $\boldsymbol{b}$, we can write

$$\begin{aligned}\mathrm{E}[b^{(\ell)}\mathrm{sgn}(y^{(m)}) \mid b^{(k)}] &= \frac{1}{2^{K-1}} \sum_{\boldsymbol{b}\in\mathcal{B}^{(k)}} b^{(\ell)} \left[1 - 2Q\left(\frac{\boldsymbol{r}_m^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}\right)\right] \\ &= \frac{-1}{2^{K-2}} \sum_{\boldsymbol{b}\in\mathcal{B}^{(k)}} b^{(\ell)} Q\left(\frac{\boldsymbol{r}_m^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}\right).\end{aligned}$$

An expression for $\mathrm{E}[b^{(m)}\mathrm{sgn}(y^{(\ell)}) \mid b^{(k)}]$ can be derived similarly.

The remaining term required to compute $\Omega_{\ell m}$ is $\mathrm{E}[\mathrm{sgn}(y^{(\ell)})\mathrm{sgn}(y^{(m)}) \mid b^{(k)}]$. Temporarily conditioning on all of the users' bits, we can write

$$\begin{aligned}\mathrm{E}[\mathrm{sgn}(y^{(\ell)})\mathrm{sgn}(y^{(m)}) \mid \boldsymbol{b}] = &+P(\{y^{(\ell)} > 0\} \cap \{y^{(m)} > 0\} \mid \boldsymbol{b}) \\ &+P(\{y^{(\ell)} < 0\} \cap \{y^{(m)} < 0\} \mid \boldsymbol{b}) \\ &-P(\{y^{(\ell)} > 0\} \cap \{y^{(m)} < 0\} \mid \boldsymbol{b}) \\ &-P(\{y^{(\ell)} < 0\} \cap \{y^{(m)} > 0\} \mid \boldsymbol{b}).\end{aligned}$$

Using the notation of [10, pp. 936], where

$$L(h,k,\rho) \triangleq \int_h^\infty \int_k^\infty g(x,y,\rho)\, dy\, dx$$

where $g(x,y,\rho)$ is the bivariate Gaussian pdf parameterized by ρ, it can be shown that

$$\begin{aligned}\mathrm{E}[\mathrm{sgn}(y^{(\ell)})\mathrm{sgn}(y^{(m)}) \mid \boldsymbol{b}] &= +L\left(\frac{-\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, \frac{-\boldsymbol{r}_m^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, \rho_{\ell m}\right) \\ &\quad +L\left(\frac{\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, \frac{\boldsymbol{r}_m^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, \rho_{\ell m}\right) \\ &\quad -L\left(\frac{-\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, \frac{\boldsymbol{r}_m^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, -\rho_{\ell m}\right) \\ &\quad -L\left(\frac{\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, \frac{-\boldsymbol{r}_m^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, -\rho_{\ell m}\right) \\ &\triangleq M\left(\frac{\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, \frac{\boldsymbol{r}_m^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, \rho_{\ell m}\right).\end{aligned}$$

Now, removing the conditioning on $\boldsymbol{b}$, we can write

$$\mathrm{E}[\mathrm{sgn}(y^{(\ell)})\mathrm{sgn}(y^{(m)}) \mid b^{(k)}] = \frac{1}{2^{K-1}} \sum_{\boldsymbol{b}\in\mathcal{B}^{(k)}} M\left(\frac{\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, \frac{\boldsymbol{r}_m^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, \rho_{\ell m}\right)$$

from which $\Omega_{\ell m}$ follows directly. Note that there is no closed form expression for $L(h,k,\rho)$ except in special cases. Computation of $\mathrm{E}[\mathrm{sgn}(y^{(\ell)})\mathrm{sgn}(y^{(m)}) \mid b^{(k)}]$ will, in general, require numerical integration.

III. $\Omega_{\ell\ell}$ for $\ell \neq k$

The results from the prior section of this appendix can be applied directly to this case, recognizing that $\mathrm{E}[(b^{(\ell)})^2 \mid b^{(k)}] = \mathrm{E}[(\mathrm{sgn}(y^{(\ell)}))^2 \mid b^{(k)}] = 1$. We can then write

$$\Omega_{\ell\ell} = 2 + \frac{1}{2^{K-3}} \sum_{\boldsymbol{b}\in\mathcal{B}^{(k)}} b^{(\ell)} Q\left(\frac{\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}\right) - \Psi_\ell^2.$$

IV. $\Phi_{\ell k}$ for $\ell \neq k$

In order to derive an exact expression for $\Phi_{\ell k}$ we will state a useful result first. Suppose that u and v are unit variance, zero mean, Gaussian random variables and that $E[uv] = \rho$. Then it can be shown via direct integration that

$$\mathrm{E}[u\mathrm{sgn}(t+v) \mid t] = \frac{2\rho}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right). \quad (10)$$

Recall that $\Phi_{\ell k} = \mathrm{E}[(b^{(\ell)} - \mathrm{sgn}(y^{(\ell)}))n^{(k)} \mid b^{(k)}]$. Since the users' bits and channel noise are assumed independent and zero mean, then $\Phi_{\ell k} = -\mathrm{E}[\mathrm{sgn}(y^{(\ell)})n^{(k)} \mid b^{(k)}]$. Conditioning temporarily on all of the users' bits, and recognize that $\mathrm{sgn}(y^{(\ell)}) = \mathrm{sgn}(y^{(\ell)}/\sigma)$ for $\sigma \neq 0$ then we can use (10) to write

$$\mathrm{E}[\mathrm{sgn}(y^{(\ell)})n^{(k)} \mid \boldsymbol{b}] = \frac{2\rho_{\ell k}}{\sqrt{2\pi}} \exp\left(\frac{-\left(\frac{-\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}\right)^2}{2}\right).$$

The conditioning on $\boldsymbol{b}$ is removed as before to write

$$\mathrm{E}[\mathrm{sgn}(y^{(\ell)})n^{(k)} \mid b^{(k)}] = \frac{\rho_{\ell k}}{\sqrt{2\pi}2^{K-2}} \sum_{\boldsymbol{b}\in\mathcal{B}^{(k)}} \exp\left(\frac{-(\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b})^2}{2\sigma^2}\right)$$

and $\Phi_{\ell k}$ follows directly.

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