SINR, Power Efficiency, and Theoretical System Capacity of Parallel Interference Cancellation

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Abstract

This paper analytically derives exact expressions for the SINR of the two-stage linear parallel interference cancellation (LPIC) and two-stage hard-decision parallel interference cancellation (HPIC) multiuser detectors in a synchronous, nonorthogonal, binary, CDMA communication system with deterministic short spreading sequences. We consider approximations to the SINR expressions that are justified in typical operating scenarios to obtain a more intuitive understanding of the SINR performance of the HPIC detector. We consider the case where a specific SINR requirement is given for each user in the system and derive expressions for the set of transmit powers necessary to meet this requirement when two-stage LPIC or HPIC detection is used. We also derive expressions for a measure of the theoretical system capacity using LPIC and HPIC detection, defined as the maximum number of users possible in a system with finite available transmit power. Numerical results are presented that compare the HPIC and LPIC detectors to the hard-decision successive interference cancellation (SIC) detector and matched filter (MF) detector. Our results suggest that HPIC detection may offer the best SINR and power efficiency performance when the number of users in the system is low to moderate and that SIC detection may offer superior performance when the number of users in the system is large.

I. Introduction

One promising technique for mitigating multiple access interference in CDMA communication systems is parallel interference cancellation (PIC). PIC was first introduced for CDMA communication systems as the multistage detector by Varanasi and Aazhang in [1] and [2]. The multistage detector was shown to have close connections to the optimum maximum likelihood detector and also to possess several desirable properties including the potential for good performance, low computational complexity, and low decision latency.

The basic idea behind all PIC detectors is that, at a given stage, decision statistics for the users in the system are formed by subtracting interference estimates (based on decision statistics from the prior stage) from the original observation. What differentiates PIC detectors is how the interference estimates are formed. Varanasi and Aazhang's multistage PIC detector uses the *hard* decisions at the output of stage $m-1$ to form the interference estimates used in stage m. More recently, Kaul and Woerner [3] proposed and analyzed an alternative multistage PIC detector which uses the *soft* decisions at the output of stage $m-1$ to form the interference estimates used in stage m. A hard decision device is not used until the final stage. Hybrid PIC detectors have also been proposed that use a linear combination of both the hard and soft decision statistics [4]. Since the pioneering work of Varanasi, Aazhang, Kaul, and Woerner, there has been an increased interest in understanding the performance of the PIC detector (see, for instance, [5], [6], [7], [8], [9], [10], [11]). For the remainder of this paper we will avoid notational confusion amongst PIC detectors by denoting the Varanasi and Aazhang detector as the hard-decision PIC (HPIC) detector and by denoting the Kaul and Woerner detector as the linear PIC (LPIC) detector.

Our contribution in this paper is an analysis of the signal to interference plus noise ratio (SINR) performance of the HPIC and LPIC detectors. We present exact expressions for the SINR of both the HPIC and LPIC detectors in the two-stage case and apply approximations where appropriate to facilitate analytical results. The results derived in this paper also lead to an analysis of the transmit power and theoretical system capacity performance of systems operating with HPIC and LPIC detection. We consider the case where each user in the CDMA communication system has a particular SINR requirement and derive an expression for the minimum transmit powers necessary to satisfy these requirements. Note that, in a nonorthogonal multiuser system such as CDMA, increasing one user's transmit power to meet their SINR requirement can also have the effect of increasing the interference seen by the other users in the system, hence lowering their SINR. We show that, for fixed SINR requirements and signature crosscorrelations, there exists a finite bound on the number of users the system can support. If the number of users in the system exceeds this bound, the total required transmit power is infinite. We call this bound the theoretical system capacity.

As a first step towards understanding the SINR performance of PIC, we derive expressions for the total required transmit power and theoretical system capacity of LPIC and HPIC detectors in the equicorrelated case where all users have identical signature sequence cross-correlations. Due to the lack of closed form results for the HPIC detector, these expressions are numerically compared to the results for SIC and MF detection provided in [12] under identical assumptions. Our results suggest that

1. LPIC detection may offer only modest performance improvements with respect to single-user MF detection in the cases considered and may actually degrade performance in some cases.

2. HPIC detection offers significant performance improvements with respect to single-user MF detection in the cases considered. HPIC detection tends to provide the best SINR and power efficiency performance, with respect to the other multiuser detectors considered in this paper, when the number of users in the system is somewhat less than the HPIC detector's system capacity.
3. SIC detection offers the best system capacity, with respect to the other multiuser detectors considered in this paper, in the cases considered. Moreover, SIC detection tends to provide the best SINR and power efficiency performance when the number of users in the system is near, or greater than, the HPIC detector's system capacity.

For the remainder of this paper we assume a synchronous CDMA multiuser communication scenario with binary signaling, nonorthogonal transmissions, and an additive white Gaussian noise channel. The communication system model is identical to the basic synchronous CDMA model described in [13]. The number of users in the system is denoted by K and all detectors considered in this paper operate on the K-dimensional matched filter bank output given by the expression

$$\boldsymbol{y}_{\mathsf{MF}} = \boldsymbol{R}\boldsymbol{A}\boldsymbol{b} + \sigma\boldsymbol{n} \tag{1}$$

where $\boldsymbol{R} \in \mathbb{R}^{K\times K}$ is a symmetric matrix of normalized user signature sequence crosscorrelations such that $\boldsymbol{R}_{kk} = 1$ for $m = 1, \ldots, K$ and $|\boldsymbol{R}_{k\ell}| \leq 1$ for all $k \neq \ell$, $\boldsymbol{A} \in \mathbb{R}^{K\times K}$ is a diagonal matrix of positive real amplitudes, $\boldsymbol{b} \in \mathbb{B}^{K\times 1}$ is the vector of i.i.d. equiprobable binary user symbols where $\mathbb{B} = \{\pm 1\}$, σ is the standard deviation of the additive channel noise, and $\boldsymbol{n} \in \mathbb{R}^{K\times 1}$ represents a matched filtered, unit variance AWGN process where $\mathrm{E}[\boldsymbol{n}] = \mathbf{0}$ and $\mathrm{E}[\boldsymbol{n}\boldsymbol{n}^\top] = \boldsymbol{R}$. The channel noise and user symbols are assumed to be independent.

II. SINR of Two-Stage PIC Detectors

Denote $\theta_{\mathsf{X}}^{(k)}$ as the SINR at the k^{th} user's output of multiuser detector X defined as

$$\theta_{\mathsf{X}}^{(k)} \triangleq \frac{\mathrm{E}[y_{\mathsf{X}}^{(k)} \mid b^{(k)}]^2}{\mathrm{var}[y_{\mathsf{X}}^{(k)} \mid b^{(k)}]}$$

where $y_{\mathsf{X}}^{(k)}$ denotes the k^{th} user's soft output from multiuser detector X prior to hard decision. In the following sections, we evaluate this expression for the two-stage LPIC and HPIC detectors in the case when the users' spreading sequences are deterministic and known.

A. LPIC Detector

The two-stage LPIC detector forms the decision statistic for the k^{th} user by the expression

$$y_{\mathsf{LPIC}}^{(k)} = a^{(k)}b^{(k)} + \sum_{\ell \neq k} \rho_{k\ell}[a^{(\ell)}b^{(\ell)} - y_{\mathsf{MF}}^{(\ell)}] + \sigma n^{(k)}.$$

Since the LPIC detector is in fact a linear detector, it is more convenient to use matrix notation to compute the SINR. Stacking the decision statistics into a K-dimensional vector, we can write

$$\boldsymbol{y}_{\mathsf{LPIC}} = \boldsymbol{y}_{\mathsf{MF}} - (\boldsymbol{R} - \boldsymbol{I})\boldsymbol{y}_{\mathsf{MF}}.$$

Using the fact that $\mathrm{E}[\boldsymbol{y}_{\mathsf{MF}} \,|\, b^{(k)}] = b^{(k)}\boldsymbol{R}\boldsymbol{A}\boldsymbol{e}_k$, it follows directly that

$$\begin{aligned}
\mathrm{E}\left[y_{\mathsf{LPIC}}^{(k)} \,|\, b^{(k)}\right] &= b^{(k)}\boldsymbol{e}_k^\top \boldsymbol{R}\boldsymbol{A}\boldsymbol{e}_k - b^{(k)}\boldsymbol{e}_k^\top(\boldsymbol{R} - \boldsymbol{I})\boldsymbol{R}\boldsymbol{A}\boldsymbol{e}_k \\
&= b^{(k)}\boldsymbol{e}_k^\top(2\boldsymbol{I} - \boldsymbol{R})\boldsymbol{R}\boldsymbol{A}\boldsymbol{e}_k
\end{aligned}$$

where $\boldsymbol{e}_k$ is a the k^{th} standard basis vector. The denominator of the SINR expression can be computed similarly and the resulting SINR of the k^{th} user's decision statistic for the LPIC detector may then be expressed without approximation as

$$\theta_{\mathsf{LPIC}}^{(k)} = \frac{\left(\boldsymbol{e}_k^\top(2\boldsymbol{I} - \boldsymbol{R})\boldsymbol{R}\boldsymbol{A}\boldsymbol{e}_k\right)^2}{\boldsymbol{e}_k^\top(2\boldsymbol{I} - \boldsymbol{R})(\boldsymbol{R}\boldsymbol{A}^2\boldsymbol{R} + \sigma^2\boldsymbol{R} - \boldsymbol{R}\boldsymbol{A}\boldsymbol{e}_k\boldsymbol{e}_k^\top\boldsymbol{A}\boldsymbol{R})(2\boldsymbol{I} - \boldsymbol{R})\boldsymbol{e}_k}. \tag{2}$$

We note that an expression for the SINR of the two-stage LPIC detector was derived in [14] for the case of random spreading sequences. Algebraic manipulation of (2) yields an alternative expression,

$$\theta_{\mathsf{LPIC}}^{(k)} = \frac{\alpha^{(k)}\left(2 - \boldsymbol{e}_k^\top\boldsymbol{R}^2\boldsymbol{e}_k\right)^2}{\sum_{\ell\neq k}\alpha^{(\ell)}\left(\boldsymbol{e}_k^\top(2\boldsymbol{I} - \boldsymbol{R})\boldsymbol{R}\boldsymbol{e}_\ell\right)^2 + \boldsymbol{e}_k^\top(2\boldsymbol{I} - \boldsymbol{R})\boldsymbol{R}(2\boldsymbol{I} - \boldsymbol{R})\boldsymbol{e}_k} \tag{3}$$

where $\alpha^{(\ell)} = (a^{(\ell)}/\sigma)^2$ is the signal to noise ratio (SNR) of the ℓ^{th} user. This last expression will be useful for the subsequent analysis in this paper where we wish to compute the set of SNRs $\{\alpha^{(k)}\}_{k=1}^K$ that achieve a desired output SINR $\{\theta^{(k)}\}_{k=1}^K$.

From the prior analysis, we note that the conditional mean of the LPIC detector's decision statistic shows a bias in the decision statistic equal to $2 - \boldsymbol{e}_k^\top \boldsymbol{R}^2 \boldsymbol{e}_k$. This bias has also been observed in [15] and [16] in the case of random spreading sequences. The relevance here is that when $\boldsymbol{e}_k^\top \boldsymbol{R}^2 \boldsymbol{e}_k > 2$, the sign of $\mathrm{E}[y_{\mathsf{LPIC}}^{(k)} \,|\, b^{(k)}]$ is not the same as that of $b^{(k)}$. In this case, the SINR expressions above do not make sense since SINR is intuitively a measure of the ratio of the decision statistic's mean squared distance from the decision boundary to its variance. When $\boldsymbol{e}_k^\top \boldsymbol{R}^2 \boldsymbol{e}_k > 2$, the mean distance from the decision boundary is actually negative and hard decisions on these decision statistics will yield errors with probability greater than 1/2. SINR analysis in this operating region in meaningless. This error probability behavior of the LPIC detector was also discussed in [17] for the general M-stage LPIC detector.

B. HPIC Detector

The two-stage HPIC detector forms the decision statistic for the k^{th} user by the expression

$$y_{\mathsf{HPIC}}^{(k)} = a^{(k)} b^{(k)} + \sum_{\ell \neq k} \rho_{k\ell} a^{(\ell)} \underbrace{[b^{(\ell)} - \mathrm{sgn}(y^{(\ell)})]}_{\epsilon^{(\ell)} \in \{-2, 0, 2\}} + \sigma n^{(k)}.$$

The SINR of the HPIC detector may then be computed as

$$\theta_{\mathsf{HPIC}}^{(k)} = \frac{\left(a^{(k)} b^{(k)} + \sum_{\ell \neq k} \rho_{k\ell} a^{(\ell)} \Psi_\ell \right)^2}{\sum_{\ell \neq k} \sum_{m \neq k} \rho_{k\ell} \rho_{km} a^{(\ell)} a^{(m)} \Omega_{\ell m} + 2\sigma \sum_{\ell \neq k} \rho_{k\ell} a^{(\ell)} \Phi_{\ell k} + \sigma^2}$$

where

$$\begin{aligned} \Psi_\ell &= \mathrm{E}[\epsilon^{(\ell)} \,|\, b^{(k)}], \\ \Omega_{\ell m} &= \mathrm{E}[\epsilon^{(\ell)} \epsilon^{(m)} \,|\, b^{(k)}] - \mathrm{E}[\epsilon^{(\ell)} \,|\, b^{(k)}] \mathrm{E}[\epsilon^{(m)} \,|\, b^{(k)}], \text{ and} \\ \Phi_{\ell k} &= \mathrm{E}[\epsilon^{(\ell)} n^{(k)} \,|\, b^{(k)}]. \end{aligned}$$

Exact expressions for Ψ, Ω, and Φ are given in the Appendix of this paper. Unfortunately, the exact expression for the SINR of the two-stage HPIC detector is unwieldy and does not lead to an intuitive understanding of its properties. Instead, we will impose the following "normal-operating" assumptions also imposed in [12] and indirectly in [13, pp. 378]:

A1. Assume that decision errors at the matched filter output of user ℓ are independent of the bits transmitted by user k ($\epsilon^{(\ell)}$ is independent of $b^{(k)}$ for all $\ell \neq k$).

A2. Assume that decision errors at the matched filter output of user ℓ are independent of decision errors at the matched filter output of user m ($\epsilon^{(\ell)}$ is independent of $\epsilon^{(m)}$ for all $\ell \neq m$).

A3. Assume that decision errors at the matched filter output of user ℓ are independent of the noise component at the soft matched filter output of user k ($\epsilon^{(\ell)}$ is independent of $n^{(k)}$ for all $\ell \neq k$).

The accuracy of these assumptions is verified numerically in Section V where the results suggest that assumptions A1–A3 are well justified unless the error probabilities at the output of the matched filter detector are high. When A1–A3 are appropriate, they imply that

$$\begin{aligned} \Psi_\ell &\approx 0 \quad \forall \ell \neq k, \\ \Omega_{\ell m} &\approx 0 \quad \forall (\ell \neq k) \neq (m \neq k), \text{ and} \\ \Phi_{\ell k} &\approx 0 \quad \forall \ell \neq k. \end{aligned}$$

The remaining term requiring calculation is $\Omega_{\ell\ell}$ which can be derived as

$$\begin{aligned} \Omega_{\ell\ell} &= \mathrm{E}[(\epsilon^{(\ell)})^2 \,|\, b^{(k)}] - \mathrm{E}[\epsilon^{(\ell)} \,|\, b^{(k)}]^2 \\ &\approx \mathrm{E}[(\epsilon^{(\ell)})^2] - 0 = 4P_{\mathsf{MF}}^{(\ell)}(\boldsymbol{\alpha}, \boldsymbol{R}) \end{aligned}$$

where $P_{\mathsf{MF}}^{(\ell)}(\boldsymbol{\alpha}, \boldsymbol{R}) = P(b^{(\ell)} \neq \mathrm{sgn}(y_{\mathsf{MF}}^{(\ell)}) \,|\, \boldsymbol{\alpha}, \boldsymbol{R})$ is the probability of error of the ℓ^{th} user's matched filter output as a function of the SNR vector $\boldsymbol{\alpha} = [\alpha^{(1)}, \ldots, \alpha^{(K)}]^\top$ and the signature sequence crosscorrelation matrix $\boldsymbol{R}$. Under these approximations, the SINR of the HPIC detector may then be written as

$$\theta_{\mathsf{HPIC}}^{(k)} = \frac{\alpha^{(k)}}{\sum_{\ell \neq k} \rho_{k\ell}^2 \alpha^{(\ell)} 4P_{\mathsf{MF}}^{(\ell)}(\boldsymbol{\alpha}, \boldsymbol{R}) + 1}. \tag{4}$$

An exact expression for the matched filter error probability $P_{\mathsf{MF}}^{(\ell)}(\boldsymbol{\alpha}, \boldsymbol{R})$ is is given in [13, pp. 113].

C. Equicorrelated Analysis

In this section, we develop additional intuition on the SINR behavior of the LPIC and HPIC detectors by examining (3) and (4) under a particular case described by the following assumptions:
A4. The signature crosscorrelations are all identical, i.e., $\rho_{k\ell} = \rho$ for all $k \neq \ell$, and
A5. The users' output SINRs are all identical, i.e., $\theta^{(k)} = \theta$ for all k.
Under these assumptions, the symmetry of the LPIC and HPIC detectors implies that the users' SNRs are also all equal, i.e., $\alpha^{(k)} = \alpha$ for all k. Applying these assumptions and their implications to (3), we can write

$$\theta_{\mathsf{LPIC}} = \frac{\alpha(1-(K-1)\rho^2)^2}{\alpha(K-1)(K-2)^2\rho^4 + 1 - (K-1)\rho^2 + (K-1)(K-2)\rho^3} \tag{5}$$

where we used the facts that

$$\begin{aligned} \boldsymbol{e}_k^\top \boldsymbol{R}^2 \boldsymbol{e}_k &= 1 + (K-1)\rho^2, \\ \boldsymbol{e}_k^\top (2\boldsymbol{I} - \boldsymbol{R})\boldsymbol{R}\boldsymbol{e}_\ell &= -(K-2)\rho^2 \ \ \forall k \neq \ell, \text{ and} \\ \boldsymbol{e}_k^\top (2\boldsymbol{I} - \boldsymbol{R})\boldsymbol{R}(2\boldsymbol{I} - \boldsymbol{R})\boldsymbol{e}_k &= 1 - (K-1)\rho^2 + (K-1)(K-2)\rho^3. \end{aligned}$$

Similarly, the SINR of the HPIC detector under assumptions A4–A5 can be computed as

$$\theta_{\mathsf{HPIC}} = \frac{\alpha}{\alpha(K-1)\rho^2 4 P_{\mathsf{MF}}(\alpha,\rho,K) + 1} \tag{6}$$

where we used the fact that the symmetry of the matched filter detector implies that the matched filter error probabilities are all equal, i.e., $P_{\mathsf{MF}}^{(k)}(\boldsymbol{\alpha}, \boldsymbol{R}) = P_{\mathsf{MF}}(\alpha,\rho,K)$ for all k.

The SINR expressions of (5) and (6) offer more intuition about the behavior of the LPIC and HPIC detectors than the general expressions developed in (3) and (4). An intuitively satisfying property that can be observed by inspection of (5) and (6) is that, for fixed α and $\rho \neq 0$, increasing K causes both detectors' SINRs to decrease to zero. A more interesting property can be observed by considering the case where K and ρ are fixed and $\alpha \to \infty$. This case represents the asymptotic SINR with infinite transmit power and represents an upper bound on the achievable SINR with finite transmit power. In this case,

$$\lim_{\alpha\to\infty} \theta_{\mathsf{LPIC}} = \frac{(1-(K-1)\rho^2)^2}{(K-1)(K-2)^2\rho^4} \tag{7}$$

and

$$\lim_{\alpha\to\infty} \theta_{\mathsf{HPIC}} \quad = \quad \frac{1}{(K-1)\rho^2 4 \lim\limits_{\alpha\to\infty} P_{\mathsf{MF}}(\alpha,\rho,K)}. \tag{8}$$

Remark 1: If $K > 2$ and $\rho \neq 0$, (7) shows that the LPIC detector's asymptotically achievable SINR is finite. When $K = 2$, the two-stage LPIC detector is actually the decorrelating detector [13] and the asymptotically achievable SINR is infinite.

Remark 2: Under the same assumptions, (8) shows that the asymptotically achievable SINR of the HPIC detector is infinite if and only if $\lim\limits_{\alpha\to\infty} P_{\mathsf{MF}}(\alpha,\rho,K) = 0$. Under assumptions A4–A5, this corresponds to the case when $(K-1)|\rho| < 1$.

III. Power Efficiency

In this section, we use the results developed in the prior section to solve the following problem for LPIC and HPIC multiuser detectors: Given a set of K users with SINR requirements $\{\theta^{(k)}\}_{k=1}^{K}$ and signature sequence crosscorrelation matrix $\boldsymbol{R}$, determine a set of transmit SNRs $\{\alpha^{(k)}\}_{k=1}^{K}$ that satisfy the requirements. Note that simply increasing any one user's power to allow that user to meet their *individual* SINR requirement will cause the SINRs of other users in the system to decrease due to the nonorthogonal multiple access interference. We proceed first with the general (not equicorrelated) case.

A. LPIC Detector

Defining $\boldsymbol{\Theta} = \mathrm{diag}(\theta^{(1)},\ldots,\theta^{(K)})$ and $\boldsymbol{\alpha} = [\alpha^{(1)},\ldots,\alpha^{(K)}]^\top$, direct algebraic manipulation of (3) allows us to write

$$\boldsymbol{D}\boldsymbol{\alpha} \quad = \quad \boldsymbol{\Theta}\left[\boldsymbol{Q}\boldsymbol{\alpha} + \boldsymbol{v}\right]$$

where $\boldsymbol{D}$ is a diagonal matrix with kk^{th} element equal to $\left(2 - \boldsymbol{e}_k^\top \boldsymbol{R}^2 \boldsymbol{e}_k\right)^2$, $\boldsymbol{Q}$ is a symmetric matrix with zeros on its diagonal and with $k\ell^{th}$ element equal to $\left(\boldsymbol{e}_k^\top(2\boldsymbol{I}-\boldsymbol{R})\boldsymbol{R}\boldsymbol{e}_\ell\right)^2$, and $\boldsymbol{v}$ is a vector with k^{th} element equal to $\boldsymbol{e}_k^\top(2\boldsymbol{I}-\boldsymbol{R})\boldsymbol{R}(2\boldsymbol{I}-\boldsymbol{R})\boldsymbol{e}_k$. Solving for $\boldsymbol{\alpha}$, we can write

$$\boldsymbol{\alpha} \quad = \quad \left[\boldsymbol{D} - \boldsymbol{\Theta}\boldsymbol{Q}\right]^{-1}\boldsymbol{\Theta}\boldsymbol{v}. \tag{9}$$

If the inverse exists in (9) then there is a unique solution for the set of users' SNRs $\{\alpha^{(k)}\}_{k=1}^{K}$ required by the LPIC detector to meet the set of target output SINRs $\{\theta^{(k)}\}_{k=1}^{K}$.

B. HPIC Detector

Defining $\mathbf{\Theta}$ and $\boldsymbol{\alpha}$ as in the prior section, we can use (4) to write

$$\left[\boldsymbol{I} - 4\mathbf{\Theta}(\mathbf{\Gamma} - \boldsymbol{I})\boldsymbol{P}_{\mathsf{MF}}(\boldsymbol{\alpha}, \boldsymbol{R})\right]\boldsymbol{\alpha} = \mathbf{\Theta}\boldsymbol{e} \tag{10}$$

where $\boldsymbol{P}_{\mathsf{MF}}(\boldsymbol{\alpha}, \boldsymbol{R})$ is the diagonal matrix with the $\ell\ell^{th}$ element equal to $P_{\mathsf{MF}}^{(\ell)}(\boldsymbol{\alpha}, \boldsymbol{R})$, $\mathbf{\Gamma}$ is the symmetric matrix of squared signature crosscorrelations with ℓk^{th} element equal to $\rho_{\ell k}^2$, and $\boldsymbol{e}$ is a K-vector with all elements equal to one. Unlike the LPIC detector, the HPIC detector's SNR requirements are not expressible in closed form due to the dependence of the matched filter detector's error probability on $\boldsymbol{\alpha}$. Numerical solution techniques are required, in general, to determine the set of SNRs $\{\alpha^{(k)}\}_{k=1}^K$ satisfying the target output SINR specification $\{\theta^{(k)}\}_{k=1}^K$.

C. Equicorrelated Analysis

Under assumptions A4–A5, we can solve (5) for α to write

$$\alpha \;=\; \frac{\theta\left[1 - (K-1)\rho^2 + (K-1)(K-2)\rho^3\right]}{(1-(K-1)\rho^2)^2 - (K-1)(K-2)^2\rho^4\theta} \tag{11}$$

for the LPIC detector. Similarly, solving (6) for α, we can write

$$\alpha \;=\; \frac{\theta}{1 - 4(K-1)\theta\rho^2 P_{\mathsf{MF}}(\alpha, \rho, K)} \tag{12}$$

for the HPIC detector. The symmetry of the LPIC and HPIC detectors under assumptions A4–A5 implies that the total SNR requirement, defined as $\sum_{k=1}^K \alpha^{(k)}$, is equal to $K\alpha$ in both cases.

Remark 1: If $\rho \neq 0$ and $\theta > 0$, (11) shows that as K increases, the denominator will decrease and begin to be dominated by the $-\rho^4\theta K^3$ term. In fact, it can be shown there is positive upper bound on K such that all K greater than this bound will cause the denominator to become negative. The same is true for the HPIC detector in (12) since $\lim_{K\to\infty} P_{\mathsf{MF}}(\alpha, \rho, K) \neq 0$ unless $\rho = 0$. These results suggest the intuitively satisfying property that, given a non-zero SINR requirement θ and non-zero signature correlations ρ, increasing K results in increasing α and that there is a finite upper bound on K for both detectors such that $\alpha < \infty$.

Remark 2: As was the case in (10), solutions to (12) for α require numerical methods in general due to the dependence of the matched filter detector's error probability on α. Nevertheless, this expression is easier to solve than (10) since it is a single nonlinear equation with a scalar valued parameter α rather than a set of simultaneous nonlinear equations with the vector valued parameter $\boldsymbol{\alpha}$.

IV. System Capacity

In this section, we use the results developed in the prior sections to solve the following problem for LPIC and HPIC multiuser detectors: Suppose that, for each $K = 1, 2, \dots$, a set of output SINR requirements $\{\theta^{(k)}\}_{k=1}^{K}$ and a signature crosscorrelation matrix $\boldsymbol{R}_K$ are given. Determine the maximum K such that the output SINR requirements can be achieved with a set of finite transmit SNRs $\{\alpha^{(k)}\}_{k=1}^{K}$. The solution to this problem can be considered the theoretical system capacity of the multiuser detector since it specifies the total number of users that can be accommodated by the multiuser detector under the finite transmit power constraint.

We will proceed directly to the equicorrelated analysis in this section since the results for the general case do not provide much useful intuition.

A. LPIC Detector Equicorrelated Analysis

To compute the theoretical system capacity for the LPIC detector, we will use the result in (7) which states that

$$\theta_{\mathsf{LPIC}} \quad < \quad \frac{(1-(K-1)\rho^2)^2}{(K-1)(K-2)^2\rho^4}$$

for finite α. Manipulation of this expression yields the desired inequality specifying the upper bound on K as

$$(K-1)^3 + \left(-2 - \frac{1}{\theta_{\mathsf{LPIC}}}\right)(K-1)^2 + \left(1 + \frac{2}{\rho^2\theta_{\mathsf{LPIC}}}\right)(K-1) - \frac{1}{\rho^4\theta_{\mathsf{LPIC}}} < 0. \tag{13}$$

It is possible to compute the roots of (13) explicitly by using the cubic equation solutions found in [18], however, the roots are quite complicated in most cases and do not yield much intuition. Given values for θ_{LPIC} and ρ, numerical solutions to this expression are easy to obtain and are presented in Section V.

It is also possible to derive a useful upper bound on the theoretical system capacity of the LPIC detector using an alternate approach. Observe that when $2 - \boldsymbol{e}_k^\top \boldsymbol{R}^2 \boldsymbol{e}_k = 0$ in (3), there exists no choice for $\alpha^{(k)}$ that achieves any positive desired SINR. In the equicorrelated case, this condition implies that $1 - (K-1)\rho^2$ must be greater than zero which leads to an upper bound on the theoretical system capacity of the two-stage LPIC detector of

$$K - 1 < \frac{1}{\rho^2}. \tag{14}$$

B. HPIC Detector Equicorrelated Analysis

The theoretical system capacity of the HPIC detector can be derived from (8). For finite α, (8) implies that

$$\theta_{\mathsf{HPIC}} < \frac{1}{(K-1)\rho^2 4 \lim_{\alpha\to\infty} P_{\mathsf{MF}}(\alpha,\rho,K)}.$$

An upper bound on K can then be written as

$$(K-1) \lim_{\alpha\to\infty} P_{\mathsf{MF}}(\alpha,\rho,K) < \frac{1}{4\rho^2\theta_{\mathsf{HPIC}}}. \tag{15}$$

Remark 1: Due to the dependence of the matched filter error probability on K, a closed form expression is not available for the theoretical system capacity of the HPIC detector. Nevertheless, numerical methods can be used to find the maximum K satisfying (15) or simple brute force techniques can be used to evaluate the left hand side of (15) for increasing integer values of K until it exceeds the right hand side.

Remark 2: As shown in Section V, the approximations used to derive (4) result in SINR expressions that tend to be somewhat optimistic when K is large. Hence, (15) should be considered an upper bound on the system capacity of the two-stage HPIC detector.

V. Numerical Results

A. Accuracy of HPIC SINR Approximations

The first numerical result in this section considers the accuracy of the approximate HPIC detection SINR results developed in (4) with respect to the exact expression for the HPIC detector's SINR as derived in the Appendix. To simplify the computations, we impose assumptions A4–A5 with signature sequence correlations set to $\rho = 1/8$ and users' SNRs set to

$10\log_{10}(\alpha) = 10$dB. Two-dimensional Gaussian quadrature numerical integration techniques are used to evaluate the exact expression for the HPIC detector's SINR over all integer values of K between 2 and 16. Two approximations for the HPIC SINR, both based on (6) but with different expressions for $P_{\mathsf{MF}}(\alpha, \rho, K)$, are also plotted for comparison. The first approximation, entitled "approx 1" uses the exact expression for the matched filter error probability given in [13, pp. 113]. The second approximation, entitled "approx 2" uses a Gaussian approximation on the matched filter error probability, i.e.,

$$P_{\mathsf{MF}}(\alpha, \rho, K) \approx Q\left(\sqrt{\theta_{\mathsf{MF}}}\right) = Q\left(\sqrt{\frac{\alpha}{(K-1)\rho^2\alpha + 1}}\right),$$

in the equicorrelated case. Although this approximation for the matched filter error probability is often inaccurate and should be used with caution, Figure 1 shows that the two approximations for the HPIC detector's SINR are nearly indistinguishable in this case. Moreover, both approximations tend to be highly accurate with respect to the exact SINR expression for values of $K \leq 10$. The approximate SINR expressions become less accurate and begin to yield increasingly optimistic values for the HPIC detector's SINR when the matched filter error probability exceeds 2E-2 at values of $K > 10$ in this example.

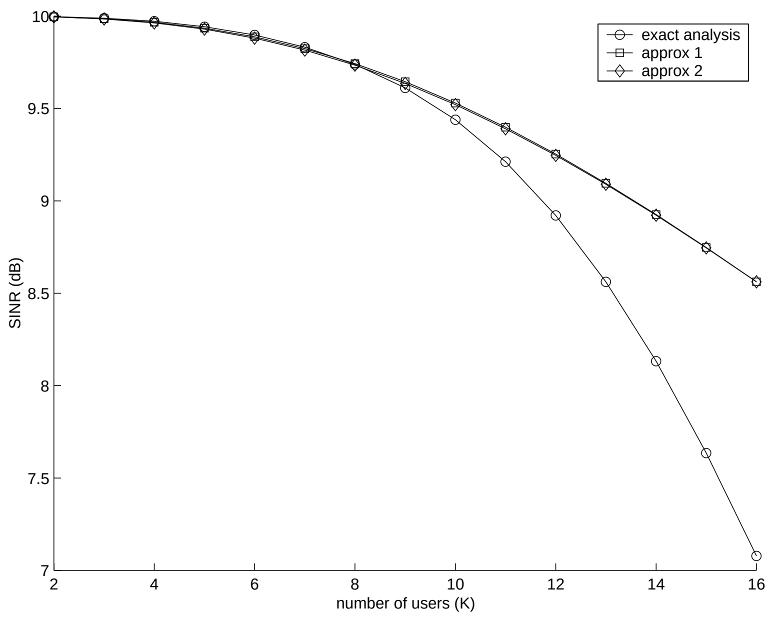

Fig. 1. Comparison of exact and approximate SINR expressions for the HPIC detector.

Figure 2 shows the HPIC detector's output distributions obtained by simulation for $K \in \{4, 8, 12, 16\}$ under the same equicorrelated operating conditions as the prior numerical example. The experimentally obtained output distributions are compared with the Gaussian distributions predicted by the exact and approximate SINR analysis in Section II-B. The theoretical and experimental results closely agree and, as seen in the prior numerical example, the approximate HPIC analysis tends to lose accuracy at larger values of K. These results also demonstrate the applicability of the SINR measure to the HPIC detector. Recent results for a large class of linear multiuser detectors (including the LPIC detector) have theoretically justified the use of the SINR measure in certain operating scenarios [19]. No such results currently exist for nonlinear multiuser detectors, hence it is not immediately clear in which cases, if any, the SINR measure is appropriate for the HPIC detector. These results experimentally suggest that the output distribution of the HPIC detector may tend to be at least near-Gaussian in some cases, including the equicorrelated case considered here, and that the SINR measure for the HPIC detector is justified in these cases.

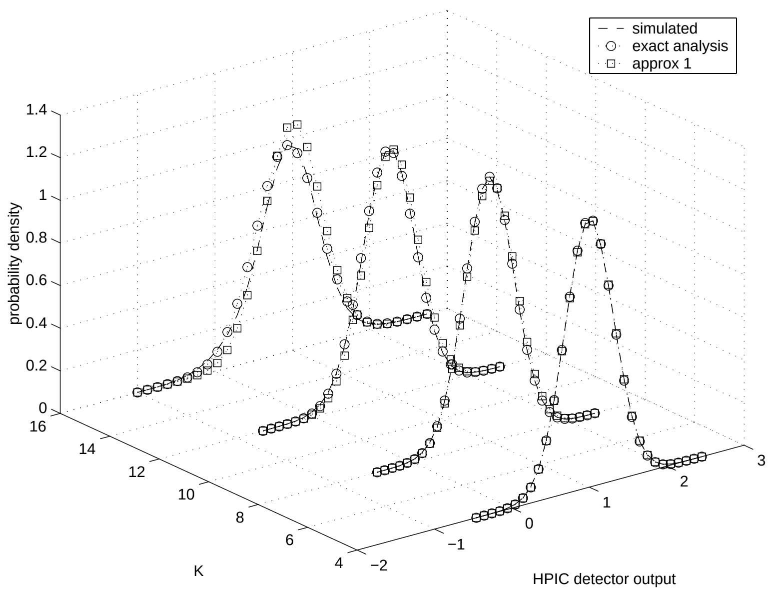

Fig. 2. Comparison of HPIC detector's output distributions obtained by simulation with Gaussian distributions predicted by exact and approximate SINR analysis.

B. Multiuser Detector Performance Comparisons

B.1 SINR Comparisons

This section compares the SINR performance of the LPIC and HPIC detectors with with the MF and SIC results given in [12] under assumption A4 and the assumption that the users' SNRs are also equal. Specifically, the SINR expressions for the LPIC and HPIC detector developed in (5) and (6), respectively, are plotted against the SINR expression for the MF detector, given as

$$\theta_{\mathsf{MF}} = \frac{\alpha}{(K-1)\rho^2\alpha + 1}$$

and the SINR expression for the hard-decision SIC detector developed in [12], given as

$$\theta_{\mathsf{SIC}}^{(k)} = \frac{\alpha}{4\sum_{\ell=1}^{k-1}\rho^2\alpha P_{\mathsf{SIC}}^{(\ell)} + (K-k)\rho^2\alpha + 1}$$

where $P_{\mathsf{SIC}}^{(\ell)}$ is the probability of error in the ℓ^{th} detected user and is approximated as $P_{\mathsf{SIC}}^{(\ell)} \approx Q\left(\sqrt{\theta_{\mathsf{SIC}}^{(\ell)}}\right)$. Since the output SINRs of the SIC detector are different for each user under the assumptions of the simulation, the arithmetic and geometric means of the SIC detector's output SINR are computed for comparison with the other multiuser detectors. The results are plotted in Figures 3 and 4.

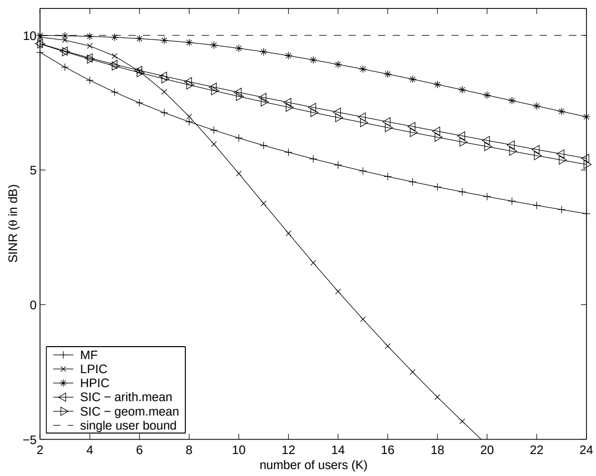

Figure 3: Multiuser detector output SINR comparison for $10\log_{10}(\alpha) = 10$dB and $\rho = 1/8$.

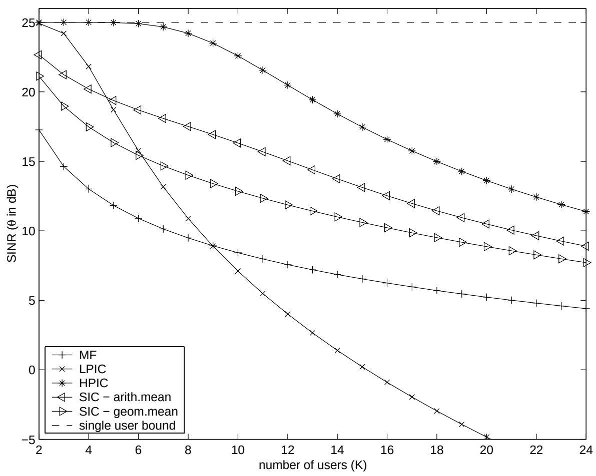

Figure 4: Multiuser detector output SINR comparison for $10\log_{10}(\alpha) = 25$dB and $\rho = 1/8$.

The results show that the HPIC detector exhibits the best performance of the tested multiuser detectors in the equicorrelated case and that the LPIC detector tends to perform very poorly for large values of K. The SIC detector exhibits better performance than the matched filter detector but does not perform as well as the HPIC detector in these examples. The accuracy results of the prior section suggest that it is reasonable to expect that the HPIC detector's output SINR, as plotted in Figures 3 and 4, is accurate for low to moderate values of K but may be somewhat optimistic at large values of K. The SINR results for the SIC detector, presented in [12], were developed under the same assumptions as A1–A3 in this paper, hence it is also reasonable to expect that the SIC detector's SINR is accurate for low to moderate values of K but may be optimistic at large values of K.

B.2 Power Efficiency Comparisons

This section compares the power efficiency of the LPIC and HPIC detectors with with the MF and SIC results given in [12] under assumptions A4–A5. The required SNR expressions for the LPIC and HPIC detectors developed in (11) and (12) are summed over all K users in the system and are plotted against the total required SNR expression for the MF detector, given as

$$\sum_{k=1}^{K} \alpha_{\mathsf{MF}}^{(k)} = \frac{K\theta}{1-(K-1)\rho^2\theta}$$

and the total required SNR expression for the hard-decision SIC detector developed in [12], given as

$$\sum_{k=1}^{K} \alpha_{\mathsf{SIC}}^{(k)} = \frac{\nu^K - 1}{\rho^2(1-4P_{\mathsf{SIC}}\nu^K)} \tag{16}$$

where $\nu = \frac{1+\rho^2\theta}{1+4P_{\mathsf{SIC}}\theta\rho^2}$ and $P_{\mathsf{SIC}} = Q(\sqrt{\theta})$ is the (identical) probability of error for each detected user of the SIC detector. We note that (16) is simplified but equivalent to the expression developed in [12]. Unlike the results of the prior section where the users were assumed to be transmitting at equal SNRs, imposing assumption A5 here causes the SIC detector to behave differently than the MF, LPIC, and HPIC detectors. Specifically, the MF, LPIC, and HPIC detectors require the users to transmit at equal SNRs to achieve equal output SINRs in the equicorrelated case, whereas the SIC detector requires the users to transmit at disparate SNRs to achieve equal output SINRs. In all cases, the results shown in this section represent the *total* power (with respect to the AWGN variance) required by the multiuser detector to meet the output SINR specification. Assumption A5 implies that all users in the system will have identical error probability for the SIC detector, hence $P_{\mathsf{SIC}}^{(\ell)} = P_{\mathsf{SIC}}$ for all ℓ.

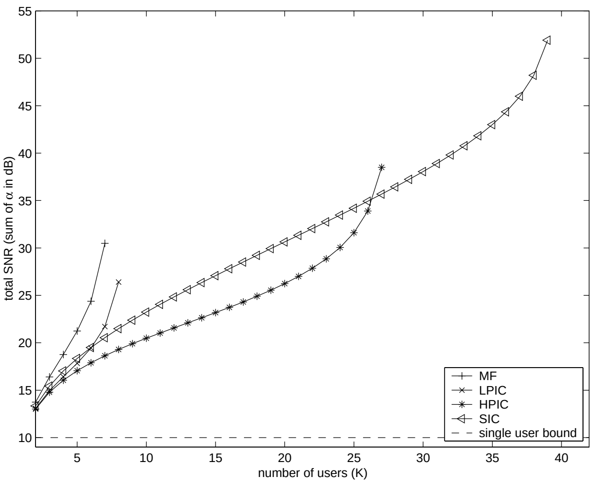

Figure 5: Multiuser detector required SNR comparison for $10\log_{10}(\theta) = 10$dB and $\rho = 1/8$.

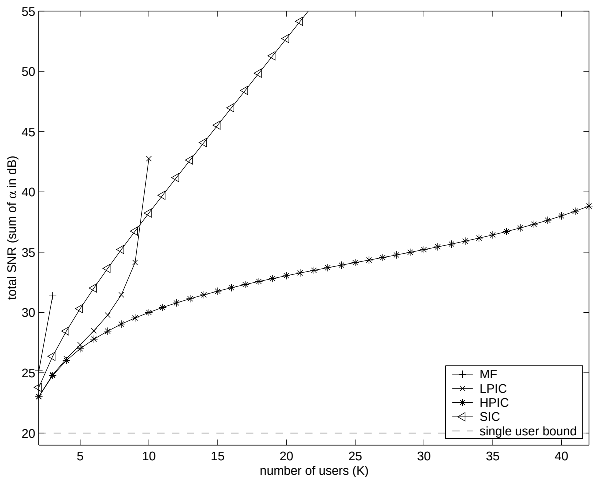

Figure 6: Multiuser detector required SNR comparison for $10\log_{10}(\theta) = 20$dB and $\rho = 1/16$.

The results in Figures 5 and 6 show that the HPIC detector exhibits better power efficiency than the other detectors, in the equicorrelated case with equal output SINRs, until K reaches a point where the matched filter decisions from the first stage become unreliable. At this point, the HPIC detector's power efficiency becomes poor and the system capacity of the HPIC detector is reached shortly thereafter. The SIC detector has relatively low power efficiency at smaller values of K but, as K approaches the value corresponding to the system capacity of the HPIC detector, the SIC detector begins to exhibit better power efficiency than the HPIC detector. The LPIC detector performs better than the matched filter and SIC detectors for small K but reaches its system capacity well before the SIC and HPIC detectors.

Unlike the results of the prior section where the HPIC detector performed better than the

SIC detector in all tested cases, this section shows that the SIC detector can outperform the HPIC detector in terms of power efficiency when K is large. The reason for this behavior is that the SIC detector, in Figures 5 and 6, selects a set of user SNRs that are disparate to achieve a set of equal output SINRs. The results shown in Figures 3 and 4 assumed equal SNRs which forced the SIC detector have disparate output SINRs. These results confirm the widely held notion that SIC detection may perform better than PIC detection when the user powers are disparate, but also suggest that PIC detection, particularly HPIC, may offer better power efficiency when the system is not operating near capacity.

B.3 Theoretical System Capacity Comparisons

This section compares the theoretical system capacity of the LPIC and HPIC detectors with with the MF and SIC results given in [12] under assumptions A4–A5. Numerical solutions to the system capacity expressions for the LPIC and HPIC detectors developed in (13) and (15) are are plotted against the theoretical system capacity of the MF detector, given as

$$K < \frac{1}{\rho^2\theta} + 1$$

and the theoretical system capacity expression for the hard-decision SIC detector developed in [12], given as

$$K < \frac{-\log(4P_{\mathsf{SIC}})}{\log \nu} \tag{17}$$

where P_{SIC} and ν are as defined in the prior section. We note that (17) is simplified but equivalent to the expression developed in [12].

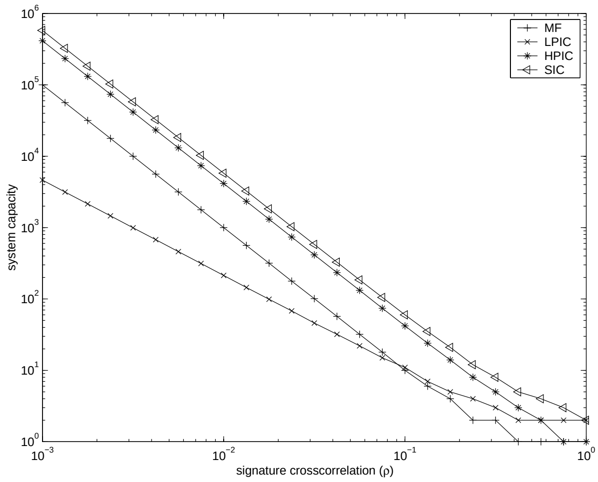

Figure 7: Multiuser detector system capacity comparison for $10\log_{10}(\theta) = 10$dB.

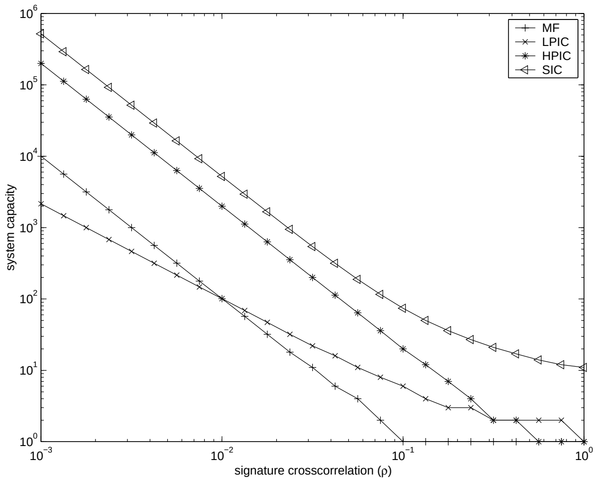

Figure 8: Multiuser detector system capacity comparison for $10\log_{10}(\theta) = 20$dB.

The results in Figures 7 and 8 show that the SIC detector exhibits better system capacity performance in the equicorrelated case given an SINR target equal for all users. The difference is more pronounced at higher SINRs and as $\rho \to 1$. The HPIC detector's system capacity is also good in these cases but, unlike the SIC detector, decreases to zero as $\rho \to 1$. The LPIC and MF detectors both performed relatively poorly in these examples with the LPIC detector performing worse than the MF detector when ρ is small and for low target SINRs. These results also confirm the results of the prior section where the SIC detector exhibited higher SNR requirements at low values of K but reached its system capacity at higher values of K than the HPIC detector.

VI. Conclusions

In this paper we derived expressions for the SINR of the LPIC and HPIC detectors and examined the implications on power efficiency and theoretical system capacity. In the case where all users have the same SINR requirement and where the signature crosscorrelations are identical between all users, we presented numerical results that suggest that the HPIC detector exhibits the best SINR performance, with respect to the LPIC, MF, and hard-decision SIC detectors, when the number of users in the system is somewhat less than the system capacity of the HPIC detector. The results suggest that LPIC detection typically offers only modest performance improvements in the cases considered and may actually degrade performance in some cases. The results also suggest that SIC detection offers the greatest system capacity in the cases considered and exhibits the best SINR performance when the number of users in the system is near, or greater than, the system capacity of the HPIC detector.

Appendix: Exact HPIC SINR Expressions

In this Appendix, we present the exact expressions for the terms used to calculate the SINR of the HPIC detector in Section II-B.

A. Ψ_ℓ for $\ell \neq k$

Recall that $\Psi_\ell = \mathrm{E}[b^{(\ell)} - \mathrm{sgn}(y^{(\ell)}) \,|\, b^{(k)}]$. Since the users' bits are assumed independent and zero mean then $\Psi_\ell = -\mathrm{E}[\mathrm{sgn}(y^{(\ell)}) \,|\, b^{(k)}]$. Conditioning temporarily on all of the users' bits, we can write

$$\begin{aligned} \mathrm{E}[\mathrm{sgn}(y^{(\ell)}) \,|\, \boldsymbol{b}] &= P(y^{(\ell)} > 0 \,|\, \boldsymbol{b}) - P(y^{(\ell)} < 0 \,|\, \boldsymbol{b}) \\ &= Q\left(\frac{-\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}\right) - \left(1 - Q\left(\frac{-\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}\right)\right) \\ &= 1 - 2Q\left(\frac{\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}\right) \end{aligned}$$

where we have used the facts that $y^{(\ell)} = \boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b} + \sigma n^{(\ell)}$ and $Q(x) + Q(-x) = 1$. To remove the conditioning on $\boldsymbol{b}$, first denote $\mathcal{B}^{(k)}$ as the set of cardinality 2^{K-1} of all possible, equiprobable, binary K-vectors with the k^{th} user's bit fixed to the known value $b^{(k)}$. Then it follows that

$$\begin{aligned} \mathrm{E}[\mathrm{sgn}(y^{(\ell)}) \,|\, b^{(k)}] &= \frac{1}{2^{K-1}} \sum_{\boldsymbol{b} \in \mathcal{B}^{(k)}} \left(1 - 2Q\left(\frac{\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}\right)\right) \\ &= 1 - \frac{1}{2^{K-2}} \sum_{\boldsymbol{b} \in \mathcal{B}^{(k)}} Q\left(\frac{\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}\right) \end{aligned}$$

and Ψ_ℓ follows directly.

B. $\Omega_{\ell m}$ for $(\ell \neq k) \neq (m \neq k)$

Recall that

$$\begin{aligned}
\Omega_{\ell m} &= \mathrm{E}[(b^{(\ell)} - \mathrm{sgn}(y^{(\ell)}))(b^{(m)} - \mathrm{sgn}(y^{(m)})) \,|\, b^{(k)}] - \Psi_\ell \Psi_m \\
&= \mathrm{E}[b^{(\ell)} b^{(m)} \,|\, b^{(k)}] - \mathrm{E}[b^{(\ell)} \mathrm{sgn}(y^{(m)}) \,|\, b^{(k)}] - \mathrm{E}[b^{(m)} \mathrm{sgn}(y^{(\ell)}) \,|\, b^{(k)}] + \\
&\quad \mathrm{E}[\mathrm{sgn}(y^{(\ell)}) \mathrm{sgn}(y^{(m)}) \,|\, b^{(k)}] - \Psi_\ell \Psi_m .
\end{aligned}$$

Since the users' bits are assumed independent and zero mean, $\mathrm{E}[b^{(\ell)} b^{(m)} \,|\, b^{(k)}] = 0$. To compute $\mathrm{E}[b^{(\ell)} \mathrm{sgn}(y^{(m)}) \,|\, b^{(k)}]$, we can temporarily condition on $\boldsymbol{b}$ to use a prior result in this Appendix to write

$$\mathrm{E}[b^{(\ell)} \mathrm{sgn}(y^{(m)}) \,|\, \boldsymbol{b}] = b^{(\ell)} \left[1 - 2Q \left(\frac{\boldsymbol{r}_m^\top \boldsymbol{A} \boldsymbol{b}}{\sigma} \right) \right].$$

Now, removing the conditioning on $\boldsymbol{b}$, we can write

$$\begin{aligned}
\mathrm{E}[b^{(\ell)} \mathrm{sgn}(y^{(m)}) \,|\, b^{(k)}] &= \frac{1}{2^{K-1}} \sum_{\boldsymbol{b} \in \mathcal{B}^{(k)}} b^{(\ell)} \left[1 - 2Q \left(\frac{\boldsymbol{r}_m^\top \boldsymbol{A} \boldsymbol{b}}{\sigma} \right) \right] \\
&= \frac{-1}{2^{K-2}} \sum_{\boldsymbol{b} \in \mathcal{B}^{(k)}} b^{(\ell)} Q \left(\frac{\boldsymbol{r}_m^\top \boldsymbol{A} \boldsymbol{b}}{\sigma} \right).
\end{aligned}$$

An expression for $\mathrm{E}[b^{(m)} \mathrm{sgn}(y^{(\ell)}) \,|\, b^{(k)}]$ can be derived similarly.

The remaining term required to compute $\Omega_{\ell m}$ is $\mathrm{E}[\mathrm{sgn}(y^{(\ell)}) \mathrm{sgn}(y^{(m)}) \,|\, b^{(k)}]$. Temporarily conditioning on all of the users' bits, we can write

$$\begin{aligned}
\mathrm{E}[\mathrm{sgn}(y^{(\ell)}) \mathrm{sgn}(y^{(m)}) \,|\, \boldsymbol{b}] = &+P(\{y^{(\ell)} > 0\} \cap \{y^{(m)} > 0\} \,|\, \boldsymbol{b}) \\
&+P(\{y^{(\ell)} < 0\} \cap \{y^{(m)} < 0\} \,|\, \boldsymbol{b}) \\
&-P(\{y^{(\ell)} > 0\} \cap \{y^{(m)} < 0\} \,|\, \boldsymbol{b}) \\
&-P(\{y^{(\ell)} < 0\} \cap \{y^{(m)} > 0\} \,|\, \boldsymbol{b}).
\end{aligned}$$

Using the notation of [18, pp. 936], where

$$L(h, k, \rho) \triangleq \int_h^\infty \int_k^\infty g(x, y, \rho)\, dy\, dx$$

where $g(x, y, \rho)$ is the bivariate Gaussian pdf parameterized by ρ, it can be shown that

$$\begin{aligned}
\mathrm{E}[\mathrm{sgn}(y^{(\ell)})\mathrm{sgn}(y^{(m)}) \,|\, \boldsymbol{b}] &= +L\left(\frac{-\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, \frac{-\boldsymbol{r}_m^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, \rho_{\ell m}\right) \\
&\quad +L\left(\frac{\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, \frac{\boldsymbol{r}_m^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, \rho_{\ell m}\right) \\
&\quad -L\left(\frac{-\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, \frac{\boldsymbol{r}_m^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, -\rho_{\ell m}\right) \\
&\quad -L\left(\frac{\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, \frac{-\boldsymbol{r}_m^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, -\rho_{\ell m}\right) \\
&\triangleq M\left(\frac{\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, \frac{\boldsymbol{r}_m^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, \rho_{\ell m}\right).
\end{aligned}$$

Now, removing the conditioning on $\boldsymbol{b}$, we can write

$$\mathrm{E}[\mathrm{sgn}(y^{(\ell)})\mathrm{sgn}(y^{(m)}) \,|\, b^{(k)}] = \frac{1}{2^{K-1}} \sum_{\boldsymbol{b}\in\mathcal{B}^{(k)}} M\left(\frac{\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, \frac{\boldsymbol{r}_m^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}, \rho_{\ell m}\right)$$

from which $\Omega_{\ell m}$ follows directly. Note that there is no closed form expression for $L(h, k, \rho)$ except in special cases. Computation of $\mathrm{E}[\mathrm{sgn}(y^{(\ell)})\mathrm{sgn}(y^{(m)}) \,|\, b^{(k)}]$ will, in general, require numerical integration.

C. $\Omega_{\ell\ell}$ for $\ell \neq k$

The results from the prior section of this appendix can be applied directly to this case, recognizing that $\mathrm{E}[(b^{(\ell)})^2 \,|\, b^{(k)}] = \mathrm{E}[(\mathrm{sgn}(y^{(\ell)}))^2 \,|\, b^{(k)}] = 1$. We can then write

$$\Omega_{\ell\ell} = 2 + \frac{1}{2^{K-3}} \sum_{\boldsymbol{b}\in\mathcal{B}^{(k)}} b^{(\ell)} Q\left(\frac{\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}\right) - \Psi_\ell^2.$$

D. $\Phi_{\ell k}$ for $\ell \neq k$

In order to derive an exact expression for $\Phi_{\ell k}$ we will state a useful result first. Suppose that u and v are unit variance, zero mean, Gaussian random variables and that $E[uv] = \rho$. Then it can be shown via direct integration that

$$\mathrm{E}[u\mathrm{sgn}(t + v) \,|\, t] = \frac{2\rho}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right). \tag{18}$$

Recall that $\Phi_{\ell k} = \mathrm{E}[(b^{(\ell)} - \mathrm{sgn}(y^{(\ell)}))n^{(k)} \,|\, b^{(k)}]$. Since the users' bits and channel noise are assumed independent and zero mean, then $\Phi_{\ell k} = -\mathrm{E}[\mathrm{sgn}(y^{(\ell)})n^{(k)} \,|\, b^{(k)}]$. Conditioning temporarily on all of the users' bits, and recognizing that $\mathrm{sgn}(y^{(\ell)}) = \mathrm{sgn}(y^{(\ell)}/\sigma)$ for $\sigma \neq 0$ then we can use (18) to write

$$\mathrm{E}[\mathrm{sgn}(y^{(\ell)})n^{(k)} \,|\, \boldsymbol{b}] \quad = \quad \frac{2\rho_{\ell k}}{\sqrt{2\pi}} \exp\left(\frac{-\left(\frac{-\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b}}{\sigma}\right)^2}{2}\right).$$

The conditioning on $\boldsymbol{b}$ is removed as before to write

$$\mathrm{E}[\mathrm{sgn}(y^{(\ell)})n^{(k)} \,|\, b^{(k)}] \quad = \quad \frac{\rho_{\ell k}}{\sqrt{2\pi}2^{K-2}} \sum_{\boldsymbol{b}\in\mathcal{B}^{(k)}} \exp\left(\frac{-(\boldsymbol{r}_\ell^\top \boldsymbol{A}\boldsymbol{b})^2}{2\sigma^2}\right)$$

and $\Phi_{\ell k}$ follows directly.

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