On the Performance of Linear Parallel Interference Cancellation

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Abstract

This paper analyzes the performance of the linear parallel interference cancellation (LPIC) multiuser detector in a synchronous multiuser communication scenario with binary signaling, nonorthogonal multiple access interference, and an additive white Gaussian noise channel. The LPIC detector has been considered in the literature lately due to its low computational complexity, potential for good performance under certain operating conditions, and close connections to the decorrelating detector. In this paper, we compare the performance of the two-stage LPIC detector to the original multistage detector proposed by Varanasi and Aazhang for CDMA systems. The general M-stage LPIC detector is compared to the conventional matched filter detector to describe operating conditions where the matched filter detector outperforms the LPIC detector in terms of error probability at any stage M. Analytical results are presented that show that the LPIC detector may exhibit divergent error probability performance under certain operating conditions and may actually yield error probabilities greater than 0.5 in some cases. Asymptotic results are presented for the case where the number of LPIC stages goes to infinity. Implications of the prior results for code division multiple access (CDMA) systems with random binary spreading sequences are discussed in the "large-system" scenario. Our results are intended to analytically corroborate the simulation evidence of other authors and to provide cautionary guidelines concerning the application of LPIC detector to CDMA communication systems.

I. Introduction

Parallel interference cancellation (PIC) is a multiuser detection [1] technique where a desired user's decision statistic is formed by subtracting an estimate of the multiple access interference from the original observation of the desired user. PIC is applicable to a wide range of asynchronous or synchronous multiuser communication systems with interfering users and is justified by the intuition that if the multiple access interference is estimated perfectly then the resulting decision statistic for the desired user contains no multiple access interference and single user performance is achieved. PIC also lends itself to a multistage structure where M concatenated PIC stages are employed to generate a set of final decision statistics. Each stage uses the prior stage's tentative decision outputs to generate new multiple access interference estimates and subtracts these interference estimates from the original observation to produce new tentative decision outputs with presumably lower multiple access interference. The first PIC detector for code division multiple access (CDMA) communication systems was derived by Varanasi and Aazhang in [2] and [3] where their PIC detector was called a multistage detector. The multistage detector was shown to have close connections to the optimum maximum likelihood detector and also to possess several desirable properties including the potential for good performance, low computational complexity, and low decision latency.

Varanasi and Aazhang's multistage detector is a particular implementation of PIC where each stage generates an output of tentative *hard* bit decisions for each user. Using the tentative hard bit decisions from the prior stage, the next stage generates its multiple access interference estimates by multiplying these tentative decisions by the corresponding user amplitudes and appropriate crosscorrelation factors. These interference estimates are then subtracted from the original observation and the result is passed through a hard decision device to form new tentative hard bit decisions for the next stage. If the multistage detector has perfect knowledge of the user amplitudes and crosscorrelation factors, and if the prior stage's bit decisions are all correct, then the multiple access interference can be perfectly cancelled and single user performance is achieved at that stage. If, on the other hand, the prior stage's outputs lead to a bit decision error for the k^{th} user then the k^{th} user's interference estimate will have the wrong sign and subtraction of this interference estimate from the original observation will result in a doubling of the interference caused by the k^{th} user on the other users' decision statistics. It has also been observed in [1, pp. 363] that Varanasi and Aazhang's multistage detector may not converge to a fixed solution as the number of stages approaches infinity. In these cases, the hard decision outputs at each stage exhibit limit cycle behavior with one or more tentative hard decisions toggling between $+1$ and -1. It was shown in [4] that this limit-cycle behavior can be explained by the multistage detector's relationship to the Hopfield neural network where limit cycles of length 2 were shown to exist in a class of special cases that includes the multistage detector [5]. Despite these shortcomings, analytical and simulation evidence suggests that the multistage detector often yields significant performance improvements over the conventional matched filter detector in typical operating scenarios. For the remainder of this paper we will avoid notational confusion with other multistage detectors by denoting the Varanasi and Aazhang multistage detector as the "hard PIC" (HPIC) detector based on the hard tentative decisions provided at the output of each stage.

More recently, Kaul and Woerner [6] proposed and analyzed an alternative PIC detector which we call the "linear PIC" (LPIC) detector. The LPIC detector is also a multistage detector but, in contrast to HPIC, each stage of the LPIC detector generates soft tentative decision outputs and a hard decision device is not used until the final stage. The soft tentative decisions of the prior stage are used to generate multiple access interference estimates for each user and these estimates are subtracted from the original observations to form new soft tentative decision outputs for the current stage. The first stage of the LPIC detector is specified to be a conventional matched filter bank. Since all operations in the generation of the final decision statistics are linear, the total operation on the original observations is linear; hence the name LPIC.

Unlike HPIC, the LPIC detector does not need to know the user amplitudes since the soft stage outputs are used as estimates for the product of each user's bit and amplitude. In fact, as will be shown later in this paper, the first stage of the LPIC detector provides unbiased estimates for this bit-amplitude product whereas the first stage of the HPIC detector outputs biased estimates. Furthermore, since the LPIC detector does not form a hard decision until the last stage, the LPIC detector does not inherently possess the interference doubling problems found in the HPIC detector. These features combined with LPIC's analytical tractability and good performance under certain operating scenarios have led to increased attention in the literature lately. The asymptotic multiuser efficiency (AME) of the M-stage LPIC detector was derived in [7] and convergence to the decorrelating detector as $M \to \infty$ was shown in [8] when the spectral radius of the user crosscorrelation matrix is less than two. The convergence behavior of both the LPIC and HPIC detectors was studied in [9]. A generalization to the LPIC detector was reported in [10] where the LPIC detector is shown to be a special case of the class of linear multiuser detectors expressible as a polynomial function of the signature crosscorrelation matrix.

This paper focuses on analyzing the behavior of the LPIC detector in a synchronous multiuser communication scenario with binary signaling, nonorthogonal transmissions, and an additive white Gaussian noise channel. Our communication system model and notation are identical to the basic synchronous CDMA model described in [1]. The number of users in the system is denoted by K and all detectors considered in this paper operate on the

K-dimensional matched filter bank output given by the expression

$$\boldsymbol{y} = \boldsymbol{R}\boldsymbol{A}\boldsymbol{b} + \sigma \boldsymbol{n} \tag{1}$$

where $\boldsymbol{R} \in \mathbb{R}^{K \times K}$ is the symmetric matrix of normalized user crosscorrelations such that $\boldsymbol{R}_{mm} = 1$ for $m = 1, \ldots, K$ and $|\boldsymbol{R}_{m\ell}| \leq 1$ for all $m \neq \ell$, $\boldsymbol{A} \in \mathbb{R}^{K \times K}$ is the diagonal matrix of positive real amplitudes, $\boldsymbol{b} \in \mathbb{B}^{K \times 1}$ is the vector of binary user symbols where $\mathbb{B} = \{\pm 1\}$, σ is the standard deviation of the channel noise, and $\boldsymbol{n} \in \mathbb{R}^{K \times 1}$ represents a matched filtered, unit variance AWGN process where $\mathrm{E}[\boldsymbol{n}] = \boldsymbol{0}$ and $\mathrm{E}[\boldsymbol{n}\boldsymbol{n}^\top] = \boldsymbol{R}$. The conventional matched filter detector forms hard decisions given by $\hat{\boldsymbol{b}}_{\mathsf{MF}} = \mathrm{sgn}(\boldsymbol{y})$. The multistage LPIC detector is given as

$$\boldsymbol{z}(m+1) = \boldsymbol{y} - (\boldsymbol{R} - \boldsymbol{I})\boldsymbol{z}(m) \qquad m = 0, 1, \ldots, M-1 \tag{2a}$$

$$\boldsymbol{z}(0) = \boldsymbol{y} \tag{2b}$$

$$\hat{\boldsymbol{b}}_{\mathsf{LPIC}} = \mathrm{sgn}(\boldsymbol{z}(M)). \tag{2c}$$

Under this notation it is evident that the two-stage[1] LPIC detector ($M = 1$) is equivalent to the *approximate decorrelator* [1] which has received some attention in the literature recently [11] due to its low computational complexity and good performance under certain operating conditions. Hence the analytical results in [11] apply here to the case when $M = 1$.

The goal of this paper is to develop a better understanding of the behavior and performance of the LPIC detector. Other authors have noted limitations in LPIC performance including the original paper by Kaul and Woerner [6] where the authors noticed that there existed conditions where interference cancellation actually degraded system performance. Since then several authors have proposed various improvements to the LPIC detector including [12], [13], [14], [15], and [16]. We do not propose to fix the LPIC detector in this paper but rather to understand it better so that we can bound the operating regions where the LPIC detector exhibits good or bad performance. In that spirit, this paper is presented as a collection of related analytical results that compare the LPIC detector to

[1] In this paper, the symbol M denotes the number of stages of interference cancellation. It is customary however to refer to an LPIC detector with one stage of interference cancellation as a “two-stage” detector, hence $M = 1$ consistently denotes the two-stage LPIC detector in this paper.

the HPIC and matched filter detectors as well as expose the asymptotic behavior of the LPIC detector as the number of stages approaches infinity.

The remainder of this paper is organized as follows. Section II compares the performance of two-stage HPIC and LPIC detectors in order to gain a better understanding of the significant performance differences between these detectors observed by other authors. Section III compares the M-stage LPIC detector to the conventional matched filter detector. Section IV analyzes operating conditions that lead to the M-stage LPIC detector exhibiting an error probability greater than 0.5. Section V develops asymptotic results on the behavior of the M-stage LPIC detector as the number of stages $M \to \infty$. Section VI examines the implications of the results in the prior sections for a CDMA communication system with random spreading sequences in the "large-system" scenario where the number of users K and the spreading gain N both approach infinity but the ratio K/N is kept constant.

II. LPIC vs. HPIC Performance Comparison

This section presents an analytical performance comparison between the two-stage HPIC and LPIC detectors. The results in this section are motivated by simulation studies of [16] and the analysis of [17] and [18] where the authors demonstrated that the two-stage HPIC detector can significantly outperform the two-stage LPIC detector in terms of error probability under a variety of operating conditions. Unfortunately, direct analysis of the two-stage HPIC detector's error probability is difficult in general since the exact HPIC error probability expressions involve K-dimensional numerical integration of the joint Gaussian probability distribution function. Rather than comparing the error probabilities of the two-stage HPIC and LPIC detectors directly, we choose to instead compare the performance of their interference estimators with the intuition that better interference estimates would tend to yield better error probability performance.

To provide a fair comparison we assume that both the LPIC and HPIC detectors use a conventional matched filter first stage. In this case, and under our synchronous system model, the two-stage LPIC and HPIC detector outputs for the k^{th} user from may be written as

$$
\begin{aligned}
z^{(k)}_{\mathsf{LPIC}} &= a^{(k)}b^{(k)} + \sum_{\ell \neq k} \rho_{k\ell} \underbrace{\left[a^{(\ell)}b^{(\ell)} - y^{(\ell)}\right]}_{\triangleq -e^{(\ell)}_{\mathsf{LPIC}}} + \sigma n^{(k)} \\
z^{(k)}_{\mathsf{HPIC}} &= a^{(k)}b^{(k)} + \sum_{\ell \neq k} \rho_{k\ell} \underbrace{\left[a^{(\ell)}b^{(\ell)} - a^{(\ell)}\mathrm{sgn}(y^{(\ell)})\right]}_{\triangleq -e^{(\ell)}_{\mathsf{HPIC}}} + \sigma n^{(k)}
\end{aligned}
$$

where $n^{(k)}$ is the k^{th} element of noise vector $\boldsymbol{n}$, $\rho_{k\ell} = \boldsymbol{R}_{k\ell}$ is the crosscorrelation coefficient between the k^{th} and ℓ^{th} users' signature waveforms, $b^{(k)}$ is the k^{th} user's element of the bit vector $\boldsymbol{b}$, and $a^{(k)} = \boldsymbol{A}_{kk}$ is the k^{th} user's amplitude. It is evident from these expressions that the fundamental difference between the two-stage HPIC and LPIC detectors is in the multiple access interference estimates. Intuitively, one would expect better estimates to generally lead to better error probability performance hence we will examine the bias and mean squared error (MSE) of the HPIC and LPIC estimators in the following analytical development.

A. LPIC Interference Estimator Performance

We can calculate the bias of the two-stage LPIC detector's multiple access interference estimator (for the ℓ^{th} user) as

$$
\begin{aligned}
\mathrm{bias}^{(\ell)}_{\mathsf{LPIC}} &= \mathrm{E}[e^{(\ell)}_{\mathsf{LPIC}} \mid b^{(\ell)}] \\
&= \mathrm{E}\left[\sum_{k \neq \ell} \rho_{\ell k} a^{(k)} b^{(k)} + \sigma n^{(\ell)}\right] \\
&= 0
\end{aligned}
$$

since $\mathrm{E}[b^{(k)}] = 0$ and $\mathrm{E}[n^{(k)}] = 0$ for all k. This shows that the matched filter outputs are conditionally unbiased estimators for the product of the ℓ^{th} user's bit and amplitude. We note that it has been observed in [12] that this unbiasedness property does not extend to additional stages of the LPIC detector and that later stages of the LPIC detector might exhibit significant bias in the multiple access interference estimates.

The MSE of the ℓ^{th} user's LPIC multiple access interference estimator can be calculated as

$$\begin{aligned}\mathrm{MSE}^{(\ell)}_{\mathsf{LPIC}} &= \mathrm{E}\left[(e^{(\ell)}_{\mathsf{LPIC}})^2 \mid b^{(\ell)}\right] \\ &= \mathrm{E}\left[\left(\sum_{k\neq\ell}\rho_{\ell k}a^{(k)}b^{(k)} + \sigma n^{(\ell)}\right)^2\right] \\ &= \sum_{k\neq\ell}\left(a^{(k)}\rho_{\ell k}\right)^2 + \sigma^2\end{aligned} \tag{3}$$

where we have used the facts that $\mathrm{E}[\boldsymbol{b}\boldsymbol{b}^\top] = \boldsymbol{I}$, $\mathrm{E}[\boldsymbol{b}\boldsymbol{n}^\top] = \boldsymbol{0}$, and $\mathrm{E}[\boldsymbol{n}\boldsymbol{n}^\top] = \boldsymbol{R}$.

B. HPIC Interference Estimator Performance

The bias of the ℓ^{th} user's multiple access interference estimator for the HPIC detector can be calculated as

$$\begin{aligned}\mathrm{bias}^{(\ell)}_{\mathsf{HPIC}} &= \mathrm{E}[e^{(\ell)}_{\mathsf{HPIC}} \mid b^{(\ell)}] \\ &= 0 \cdot P(\mathrm{sgn}(y^{(\ell)}) = b^{(\ell)}) - 2a^{(\ell)}b^{(\ell)} \cdot P(\mathrm{sgn}(y^{(\ell)}) \neq b^{(\ell)}) \\ &= -2a^{(\ell)}b^{(\ell)}P(\mathrm{sgn}(y^{(\ell)}) \neq b^{(\ell)})\end{aligned}$$

where $P(\mathrm{sgn}(y^{(\ell)}) \neq b^{(\ell)}) = P^{(\ell)}_{\mathsf{MF}}$ is the matched filter detector's probability of bit error for user ℓ given by the expression in [1] as

$$P(\mathrm{sgn}(y^{(\ell)}) \neq b^{(\ell)}) = \frac{1}{2^{K-1}} \sum_{\substack{b^{(\ell)}=1 \\ b^{(k)}\in\{\pm1\}\ \forall k\neq\ell}} Q\left(\frac{a^{(\ell)}b^{(\ell)} + \sum_{k\neq\ell}\rho_{\ell k}a^{(k)}b^{(k)}}{\sigma}\right)$$

where $Q(x) \triangleq \int_x^\infty e^{-t^2/2}\,dt$. We observe that the ℓ^{th} user's HPIC multiple access interference estimator is biased unless $P(\mathrm{sgn}(y^{(\ell)}) \neq b^{(\ell)}) = 0$.

The MSE of the ℓ^{th} user's HPIC multiple access interference estimator can be calculated as

$$\begin{aligned}\mathrm{MSE}^{(\ell)}_{\mathsf{HPIC}} &= \mathrm{E}\left[(e^{(\ell)}_{\mathsf{HPIC}})^2 \mid b^{(\ell)}\right] \\ &= (a^{(\ell)})^2\mathrm{E}\left[|\mathrm{sgn}(y^{(\ell)}) - b^{(\ell)}|^2\right] \\ &= (a^{(\ell)})^2[0 \cdot P(\mathrm{sgn}(y^{(\ell)}) = b^{(\ell)}) + 4 \cdot P(\mathrm{sgn}(y^{(\ell)}) \neq b^{(\ell)})] \\ &= (a^{(\ell)})^2 4P(\mathrm{sgn}(y^{(\ell)}) \neq b^{(\ell)}).\end{aligned} \tag{4}$$

We note that both the bias and MSE of the HPIC multiple access interference estimator are proportional to the probability of bit error from the first (matched filter) stage.

C. LPIC vs. HPIC Performance Comparison

An exact analytical comparison of $\mathrm{MSE}_{\mathsf{LPIC}}^{(\ell)}$ and $\mathrm{MSE}_{\mathsf{HPIC}}^{(\ell)}$ is difficult due to the sum of Q functions involved in the evaluation of (4). An explanation for the significant performance difference between the HPIC and LPIC detector seen in the simulation results of [16] is possible if we resort to a Gaussian approximation for the multiple access interference (e.g., see [19]). Even though this approximation is not valid under all circumstances (see [20]), its use in this case provides some insight into the relative performance of the HPIC and LPIC multiple access interference estimators in the absence of more exact methods. Moreover, the result presented in the following proposition is shown in Section VI-A to be asymptotically exact in the case of large CDMA systems with random spreading sequences.

Under the Gaussian approximation assumption, the multiple access interference is assumed to be well-modeled as a Gaussian random variable and the probability of bit error for user ℓ can be written as

$$P(\mathrm{sgn}(y^{(\ell)}) \neq b^{(\ell)}) \approx Q\left(\frac{a^{(\ell)}}{\sqrt{\sum_{k\neq\ell}\left(a^{(\ell)}\rho_{\ell k}\right)^2+\sigma^2}}\right) = Q\left(\frac{a^{(\ell)}}{\sqrt{\mathrm{MSE}_{\mathsf{LPIC}}^{(\ell)}}}\right)$$

hence

$$\mathrm{MSE}_{\mathsf{HPIC}}^{(\ell)} \approx \mathrm{AMSE}_{\mathsf{HPIC}}^{(\ell)} = 4(a^{(\ell)})^2 Q\left(\frac{a^{(\ell)}}{\sqrt{\mathrm{MSE}_{\mathsf{LPIC}}^{(\ell)}}}\right) \tag{5}$$

where $\mathrm{AMSE}_{\mathsf{HPIC}}^{(\ell)}$ denotes the *approximate* MSE of the ℓ^{th} user's interference estimate with the HPIC detector. With this development we can prove the following proposition.

Proposition 1: For arbitrary $\boldsymbol{R}$, σ, $\boldsymbol{A}$, K, and ℓ, $\mathrm{MSE}_{\mathsf{LPIC}}^{(\ell)} > \mathrm{AMSE}_{\mathsf{HPIC}}^{(\ell)}$.

Proof: Let $x = a^{(\ell)}/\sqrt{\mathrm{MSE}_{\mathsf{LPIC}}^{(\ell)}}$. Then $x > 0$ since $a^{(\ell)} > 0$ and $\mathrm{MSE}_{\mathsf{LPIC}}^{(\ell)} > 0$. An upper bound on the Q function for $x > 0$ is given in [1] where

$$Q(x) < \frac{1}{\sqrt{2\pi}x}\exp\left(-\frac{x^2}{2}\right).$$

Then

$$\mathrm{AMSE}_{\mathsf{HPIC}}^{(\ell)} < 4a^{(\ell)}\sqrt{\frac{\mathrm{MSE}_{\mathsf{LPIC}}^{(\ell)}}{2\pi}}\exp\left(-\frac{x^2}{2}\right)$$

hence it suffices to show

$$\frac{4x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) < 1$$

for $x > 0$ in order to prove the proposition. Since both sides of the inequality are positive we can take the natural logarithm to write

$$\ln(x) + \ln\left(\frac{4}{\sqrt{2\pi}}\right) - \frac{x^2}{2} < 0$$

but $\ln(x) < x - 1$ for all $x > 0$ hence

$$\ln(x) + \ln\left(\frac{4}{\sqrt{2\pi}}\right) - \frac{x^2}{2} < x - 1 + \ln\left(\frac{4}{\sqrt{2\pi}}\right) - \frac{x^2}{2}. \tag{6}$$

The discriminant of the right hand side of (6) is given by

$$1 - 4\left[\frac{1}{2} - \frac{1}{2}\ln\left(\frac{4}{\sqrt{2\pi}}\right)\right] = -1 + 2\ln\left(\frac{4}{\sqrt{2\pi}}\right)$$

which is strictly less than zero, hence the quadratic equation in (6) has no real roots. This implies that (6) is either always less than zero or greater than zero. Inspection of (6) shows that it is always less than zero, hence $\mathrm{MSE}^{(\ell)}_{\mathsf{LPIC}} > \mathrm{AMSE}^{(\ell)}_{\mathsf{HPIC}}$. ∎

As a numerical example of the interference estimator performance, consider a multiuser communication system with $K = 6$ equipower, equicorrelated users such that $\rho_{k\ell} = \rho$ for all $k \neq \ell$. The exact and approximate interference estimator MSE performance for the two-stage LPIC and HPIC detector is shown in Figure 1 over a range of typical SNR values for several values of ρ. Note that the approximate HPIC interference estimator MSE ($\mathrm{AMSE}^{(k)}_{\mathsf{HPIC}}$) is quite accurate in all of the cases shown and is nearly indistinguishable from the exact HPIC interference estimator MSE ($\mathrm{MSE}^{(k)}_{\mathsf{HPIC}}$) in the cases where $\rho = 0.2$ and $\rho = 0.5$. Moreover, these cases demonstrate the superiority of the HPIC interference estimator in terms of MSE and give some feeling for its relative performance with respect to the LPIC interference estimator.

III. Comparison to the Matched Filter Detector

The goal of this section is to show that the error probability of the matched filter detector is lower than that of the M-stage LPIC detector when the desired user's amplitude exceeds a finite threshold parameterized by M. It was shown in [1, pp. 251-5] that the matched

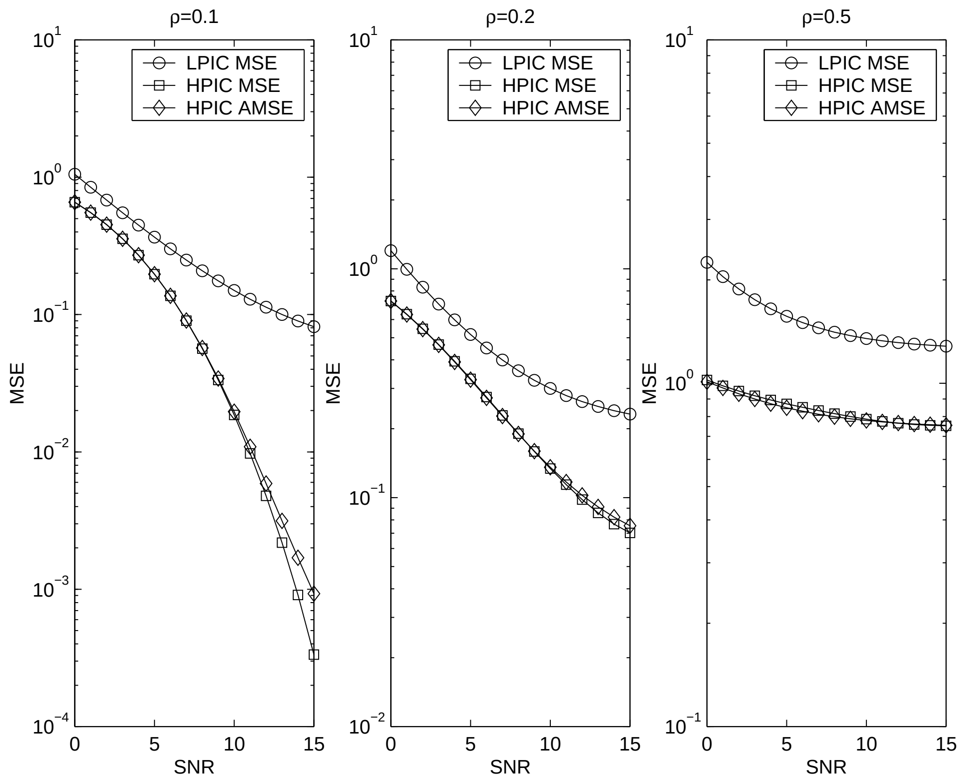

Fig. 1. MSE of two-stage LPIC and HPIC estimators for 6 equipower, equicorrelated users.

filter detector can perform better than the decorrelating detector in the low SNR case as well as the two-user case when the desired user's amplitude is much larger than the interfering user. More general results comparing the matched filter to the decorrelating and MMSE detectors have recently been obtained by Moustakides and Poor in [21]. Here, we use a similar method of proof for the LPIC detector but we also derive a closed form expression for a sufficient threshold on the desired user's amplitude parameterized by the number of interference cancellation stages.

Proposition 2: Denote the error probability for the k^{th} user of the M-stage LPIC and matched filter detectors as $P^{(k)}_{\mathsf{LPIC}}(M)$ and $P^{(k)}_{\mathsf{MF}}$, respectively. Given an arbitrary fixed desired user k, an LPIC detector with $M \geq 1$ stages of interference cancellation, a signature crosscorrelation matrix $\boldsymbol{R} \neq \boldsymbol{I}$, and noise standard deviation $\sigma > 0$, then $P^{(k)}_{\mathsf{LPIC}}(M) > P^{(k)}_{\mathsf{MF}}$ if

$$a^{(k)} > \frac{\sum_{\ell \neq k} a^{(\ell)} \left| \theta_{k\ell}(M) - \rho_{k\ell} \right|}{1 - \theta_{kk}(M)} \tag{7}$$

for

$$\theta_{k\ell}(M) \triangleq \frac{\boldsymbol{e}_k^\top \boldsymbol{L}(M)\boldsymbol{R}\boldsymbol{e}_\ell}{\sqrt{\boldsymbol{e}_k^\top \boldsymbol{L}(M)\boldsymbol{R}\boldsymbol{L}(M)\boldsymbol{e}_k}}. \tag{8}$$

Proof: For nonzero noise power, the probability of a decision error for the k^{th} user of an arbitrary linear detector F can be expressed as

$$P_{\mathsf{F}}^{(k)} = \frac{1}{2^{K-1}} \sum_{\boldsymbol{b}\in\mathcal{B}_k} Q\left(\frac{\boldsymbol{f}^{(k)\top}\boldsymbol{R}\boldsymbol{A}\boldsymbol{b}}{\sigma\sqrt{\boldsymbol{f}^{(k)\top}\boldsymbol{R}\boldsymbol{f}^{(k)}}}\right)$$

where $\mathcal{B}_k$ is the set of all possible bit vectors such that $b^{(k)} = +1$ and $b^{(\ell)} \in \{\pm 1\}$ for all $\ell \neq k$ and $\boldsymbol{f}^{(k)} \in \mathbb{R}^{K\times 1}$ denotes the effective linear operation on the matched filter bank outputs to form the decision statistic for user k. The matched filter detector is given as $\boldsymbol{f}^{(k)} = \boldsymbol{e}_k$. The M-stage LPIC detector can be written as

$$\boldsymbol{z}(M) = \sum_{m=0}^{M} (\boldsymbol{I} - \boldsymbol{R})^m \boldsymbol{y} = \boldsymbol{L}(M)\boldsymbol{y}$$

hence $\boldsymbol{f}^{(k)} = \boldsymbol{L}(M)\boldsymbol{e}_k$. Since $Q(x)$ is a monotonically decreasing function in x,

$$\frac{\boldsymbol{e}_k^\top \boldsymbol{L}(M)\boldsymbol{R}\boldsymbol{A}\boldsymbol{b}}{\sigma\sqrt{\boldsymbol{e}_k^\top \boldsymbol{L}(M)\boldsymbol{R}\boldsymbol{L}(M)\boldsymbol{e}_k}} < \frac{\boldsymbol{e}_k^\top \boldsymbol{R}\boldsymbol{A}\boldsymbol{b}}{\sigma\sqrt{\boldsymbol{e}_k^\top \boldsymbol{R}\boldsymbol{e}_k}} \quad \forall \boldsymbol{b} \in \mathcal{B}_k \tag{9}$$

implies that $P_{\mathsf{LPIC}}^{(k)}(M) > P_{\mathsf{MF}}^{(k)}$. We note that this is a sufficient condition and the converse is not necessarily true. Observing that $\boldsymbol{e}_k^\top \boldsymbol{R}\boldsymbol{e}_k = 1$, $\boldsymbol{A}\boldsymbol{e}_k = a^{(k)}\boldsymbol{e}_k$, $\boldsymbol{b} = \boldsymbol{e}_k + \sum_{\ell\neq k} b^{(\ell)}\boldsymbol{e}_\ell$, and cancelling σ from both sides of the inequality, we can rewrite (9) as

$$a^{(k)}\theta_{kk}(M) + \sum_{\ell\neq k} a^{(\ell)} b^{(\ell)} \theta_{k\ell}(M) < a^{(k)} + \sum_{\ell\neq k} a^{(\ell)} b^{(\ell)} \rho_{k\ell} \quad \forall \boldsymbol{b} \in \mathcal{B}_k \tag{10}$$

for $\theta_{k\ell}$ defined by (8). Using the Schwarz inequality and the fact that $\boldsymbol{R}$ is nonnegative definite, we note that

$$\begin{aligned}
\boldsymbol{e}_k^\top \boldsymbol{L}(M)\boldsymbol{R}\boldsymbol{e}_k &= \boldsymbol{e}_k^\top \boldsymbol{L}(M)\boldsymbol{R}^{1/2}\boldsymbol{R}^{1/2}\boldsymbol{e}_k \\
&\leq \sqrt{\boldsymbol{e}_k^\top \boldsymbol{L}(M)\boldsymbol{R}\boldsymbol{L}(M)\boldsymbol{e}_k}\sqrt{\boldsymbol{e}_k^\top \boldsymbol{R}\boldsymbol{e}_k} \\
&= \sqrt{\boldsymbol{e}_k^\top \boldsymbol{L}(M)\boldsymbol{R}\boldsymbol{L}(M)\boldsymbol{e}_k}
\end{aligned}$$

with equality if and only if $\boldsymbol{L}(M) = \alpha\boldsymbol{I}$ or if and only if $\boldsymbol{R} = \boldsymbol{I}$. In the case where $\boldsymbol{R} = \boldsymbol{I}$ the users' signatures are all mutually orthogonal and the LPIC detector is identical to the matched filter detector. Since the proposition assumes that $\boldsymbol{R} \neq \boldsymbol{I}$, this implies that $0 < \theta_{kk} < 1$ and we can rearrange the terms in (10) to write

$$a^{(k)} > \frac{\sum_{\ell \neq k} a^{(\ell)} b^{(\ell)} \left[\theta_{k\ell}(M) - \rho_{k\ell}\right]}{1 - \theta_{kk}(M)} \quad \forall \boldsymbol{b} \in \mathcal{B}_k.$$

We can remove the dependence on $\boldsymbol{b}$ from this expression by exploiting the binary nature of its elements to maximize the right hand side of the inequality by setting

$$b^{(\ell)} = \operatorname{sgn}\left(\theta_{k\ell}(M) - \rho_{k\ell}\right) \quad \forall \ell \neq k$$

from which (7) follows directly. ■

We note that the proof does not rely on the structure of the LPIC detector and the above analysis applies to any linear detector that is not a function of the user amplitudes including the decorrelating detector. Computation of the threshold on $a^{(k)}$ is, however, dependent on the particular linear detector. We also note that the derived threshold is not necessary but sufficient and is likely to be loose in the sense that values of $a^{(k)}$ significantly less than the threshold may also cause the LPIC detector to exhibit a higher probability of bit error than the matched filter detector.

As a numerical example, consider a communication system with $K = 3$ users with $a^{(k)}/\sigma = 2$ for $k \in \{2, 3\}$. Suppose that the normalized user signature crosscorrelation matrix is given by

$$\boldsymbol{R} = \frac{1}{5} \begin{bmatrix} 5 & 3 & 1 \\ 3 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}.$$

Computation of the amplitude threshold under these conditions yields values approximately equal to $3.04, 4.40, 3.12,$ and 2.97 for the cases when $M = 1, 2, 3,$ and 100, respectively. A plot of the error probabilities in Figure 2 shows that the actual crossover points occur at approximately $2.8, 4.4, 2.8,$ and 2.2, respectively. It also interesting to note that in the $M = 100$ case[2] $\boldsymbol{L}(100) \approx \boldsymbol{R}^{-1}$, yet the LPIC detector exhibits better error probability performance for $M \in \{1, 2, 3\}$. This example illustrates that the LPIC detector's performance is unfortunately not monotonic in M in general and that the error probability

[2] $\lim_{M \to \infty} \boldsymbol{L}(M) = \boldsymbol{R}^{-1}$ since the spectral radius of $\boldsymbol{R}$ is equal to 1.6 in this example.

performance of the LPIC detector may actually degrade as M increases even in cases when the LPIC detector is known to converge to the decorrelating detector.

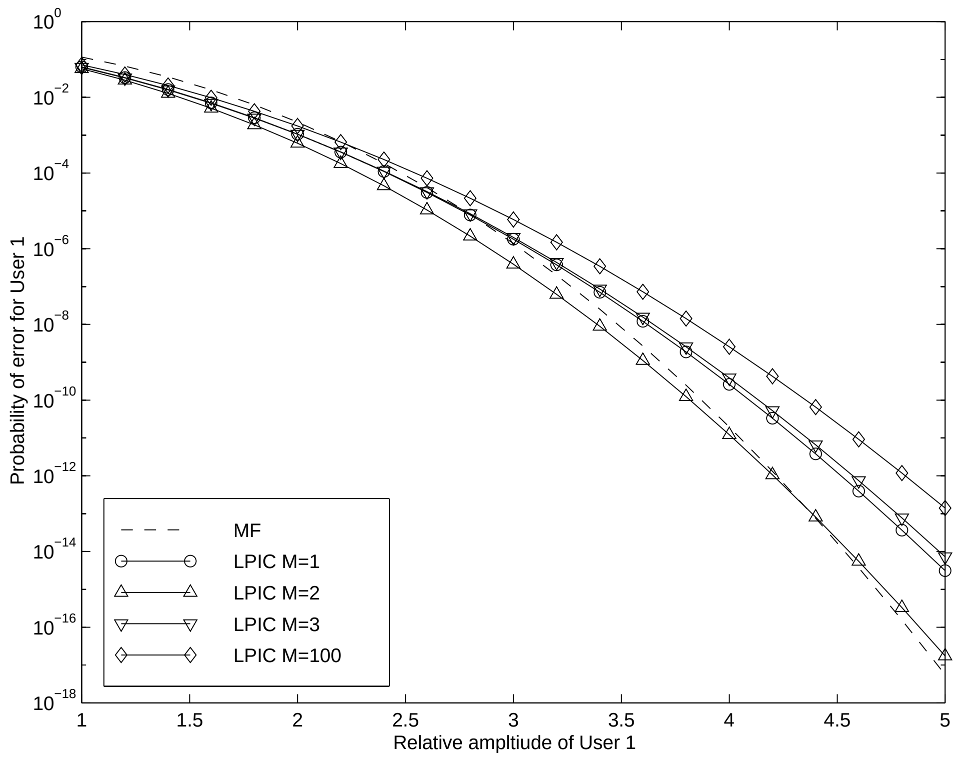

Fig. 2. Example of error probabilities $P^{(1)}_{\mathsf{LPIC}}(M)$ and $P^{(1)}_{\mathsf{MF}}$.

IV. LPIC Error Probability Divergence for Finite M

In this section, we derive an explicit description of a class of signature crosscorrelation matrices under which the LPIC detector exhibits an error probability $P^{(k)}_{\mathsf{LPIC}}(M) > 0.5$ for arbitrary odd values of M.[3] This behavior is in contrast to the matched filter detector which never exhibits an error probability greater than 0.5 under any operating conditions within the scope of the K-user, synchronous, binary system model. We make this claim more precise with the following proposition.

Proposition 3: For an arbitrary fixed desired user k in a system with $K \geq 3$ users,

[3]We note that if the detector is aware of the fact that its binary decisions have error probability greater than 0.5 then a simple sign change on the decisions would yield an error probability of less than 0.5. In this context, this section describes a subset of the operating regions where this misperformance occurs so as to alert the detector when such a bit-flipping strategy might be beneficial.

an LPIC detector with $M \geq 1$ stages of interference cancellation where M is odd, and equicorrelated users such that $\boldsymbol{R}_{ij} = \rho \; \forall i \neq j$, then $P^{(k)}_{\mathsf{LPIC}}(M) > 0.5$ if

$$\left(\frac{K}{(K-1)^{M+1} + (K-1)}\right)^{\frac{1}{M+1}} < \rho \leq 1. \tag{11}$$

Proof: We can write the argument of the Q function for the LPIC detector's error probability expression as

$$\frac{\boldsymbol{e}_k^\top \boldsymbol{L}(M)\boldsymbol{R}\boldsymbol{A}\boldsymbol{b}}{\sigma\sqrt{\boldsymbol{e}_k^\top \boldsymbol{L}(M)\boldsymbol{R}\boldsymbol{L}(M)\boldsymbol{e}_k}} = \underbrace{\frac{a^{(k)}\boldsymbol{e}_k^\top \boldsymbol{L}(M)\boldsymbol{R}\boldsymbol{e}_k}{\sigma\sqrt{\boldsymbol{e}_k^\top \boldsymbol{L}(M)\boldsymbol{R}\boldsymbol{L}(M)\boldsymbol{e}_k}}}_{\triangleq \alpha_k(M)} + \underbrace{\frac{\boldsymbol{e}_k^\top \boldsymbol{L}(M)\boldsymbol{R}\boldsymbol{A}\boldsymbol{d}}{\sigma\sqrt{\boldsymbol{e}_k^\top \boldsymbol{L}(M)\boldsymbol{R}\boldsymbol{L}(M)\boldsymbol{e}_k}}}_{\triangleq \beta_k(\boldsymbol{d},M)} \tag{12}$$

where $\boldsymbol{d} = \boldsymbol{b} - \boldsymbol{e}_k$. The error probability of the M-stage LPIC detector may then be expressed as

$$P^{(k)}_{\mathsf{LPIC}}(M) = \frac{1}{2^{K-1}} \sum_{\boldsymbol{d} \in \mathcal{D}_k} Q\left(\alpha_k(M) + \beta_k(\boldsymbol{d}, M)\right) \tag{13}$$

where $\mathcal{D}_k$ is the set of all vectors such that the k^{th} element $d^{(k)} = 0$ and $d^{(\ell)} \in \{\pm 1\}$ for all $\ell \neq k$. Recognizing that $\boldsymbol{d} \in \mathcal{D}_k$ implies that $-\boldsymbol{d} \in \mathcal{D}_k$ and that $\beta_k(-\boldsymbol{d}, M) = -\beta_k(\boldsymbol{d}, M)$ we can rewrite (13) as

$$P^{(k)}_{\mathsf{LPIC}}(M) = \frac{1}{2^K} \sum_{\boldsymbol{d} \in \mathcal{D}_k} Q\left(\alpha_k(M) + \beta_k(\boldsymbol{d}, M)\right) + Q\left(\alpha_k(M) - \beta_k(\boldsymbol{d}, M)\right).$$

Suppose temporarily that $\alpha_k(M) < 0$. Since $Q(x)$ is monotonically decreasing in x then

$$Q\left(\alpha_k(M) + \beta_k(\boldsymbol{d}, M)\right) + Q\left(\alpha_k(M) - \beta_k(\boldsymbol{d}, M)\right) > Q\left(\beta_k(\boldsymbol{d}, M)\right) + Q\left(-\beta_k(\boldsymbol{d}, M)\right) = 1$$

and it follows directly that

$$\frac{1}{2^K} \sum_{\boldsymbol{d} \in \mathcal{D}_k} Q\left(\alpha_k(M) + \beta_k(\boldsymbol{d}, M)\right) + Q\left(\alpha_k(M) - \beta_k(\boldsymbol{d}, M)\right) > \frac{1}{2^K} \sum_{\boldsymbol{d} \in \mathcal{D}_k} 1 = \frac{2^{K-1}}{2^K} = \frac{1}{2}.$$

Hence it is sufficient to show that if ρ satisfies (11) then $\alpha_k(M) < 0$ in the equicorrelated case. Since $a^{(k)}/\sigma > 0$ and $\sqrt{\boldsymbol{e}_k^\top \boldsymbol{L}(M)\boldsymbol{R}\boldsymbol{L}(M)\boldsymbol{e}_k} > 0$ then $\alpha_k(M) < 0$ if and only if $\boldsymbol{e}_k^\top \boldsymbol{L}(M)\boldsymbol{R}\boldsymbol{e}_k < 0$. We note that for the matched filter detector $\boldsymbol{L}(M) = \boldsymbol{I}$ hence $\alpha_k(M) = a^{(k)}/\sigma > 0$ for all $\boldsymbol{R}$. This justifies our earlier claim that the matched filter detector cannot have an error probability greater than 0.5.

Returning to the LPIC detector, we wish to show that $\boldsymbol{e}_k^\top \boldsymbol{L}(M)\boldsymbol{R}\boldsymbol{e}_k < 0$ for ρ satisfying (11) in the equicorrelated case. To show this, recall that the multistage LPIC detector may be written as

$$\boldsymbol{L}(M) = \sum_{m=0}^{M} (\boldsymbol{I} - \boldsymbol{R})^m.$$

Let $\boldsymbol{I} - \boldsymbol{R} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$ where $\boldsymbol{V}$ is a matrix with columns representing the eigenvectors of $(\boldsymbol{I} - \boldsymbol{R})$ and $\boldsymbol{\Lambda}$ is a diagonal matrix of corresponding eigenvalues. Then

$$\boldsymbol{L}(M) = \boldsymbol{V} \left[\sum_{m=0}^{M} \boldsymbol{\Lambda}^m \right] \boldsymbol{V}^{-1}.$$

It can be shown that each eigenvector of $(\boldsymbol{I} - \boldsymbol{R})$ is also an eigenvector of $\boldsymbol{R}$ and that if λ is an eigenvalue of $(\boldsymbol{I} - \boldsymbol{R})$ then $1 - \lambda$ is an eigenvalue of $\boldsymbol{R}$. Using these facts, we can write

$$\begin{aligned}
\boldsymbol{L}(M)\boldsymbol{R} &= \boldsymbol{V} \left[\sum_{m=0}^{M} \boldsymbol{\Lambda}^m \right] (\boldsymbol{I} - \boldsymbol{\Lambda})\boldsymbol{V}^{-1} \\
&= \boldsymbol{V} \left[\sum_{m=0}^{M} \boldsymbol{\Lambda}^m - \sum_{m=0}^{M} \boldsymbol{\Lambda}^{m+1} \right] \boldsymbol{V}^{-1} \\
&= \boldsymbol{V} \left[\boldsymbol{I} - \boldsymbol{\Lambda}^{M+1} \right] \boldsymbol{V}^{-1} \\
&= \boldsymbol{I} - \boldsymbol{V}\boldsymbol{\Lambda}^{M+1}\boldsymbol{V}^{-1}
\end{aligned}$$

hence

$$\boldsymbol{e}_k^\top \boldsymbol{L}(M)\boldsymbol{R}\boldsymbol{e}_k = 1 - \boldsymbol{e}_k^\top \boldsymbol{V}\boldsymbol{\Lambda}^{M+1}\boldsymbol{V}^{-1}\boldsymbol{e}_k. \tag{14}$$

Applying the equicorrelated assumption, it can be shown that $\boldsymbol{I} - \boldsymbol{R}$ has one eigenvalue equal to $(1 - K)\rho$ and $K - 1$ eigenvalues equal to ρ. Furthermore, it can be shown that $\boldsymbol{V}$ can be written in the form

$$\boldsymbol{V} = \left[\begin{array}{cccc} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \ldots & \boldsymbol{v}_K \end{array} \right]$$

where the normalized eigenvectors are given by

$$
\begin{aligned}
\boldsymbol{v}_1 &= \frac{1}{\sqrt{K}} \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \end{bmatrix}^\top \\
\boldsymbol{v}_2 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 \end{bmatrix}^\top \\
\boldsymbol{v}_3 &= \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & -2 & 0 & \dots & 0 \end{bmatrix}^\top \\
&\vdots \\
\boldsymbol{v}_K &= \frac{1}{\sqrt{(K-1)+(K-1)^2}} \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & -(K-1) \end{bmatrix}^\top
\end{aligned}
$$

and where $\boldsymbol{v}_1$ is the eigenvector corresponding to the unique eigenvalue. We have used the normalized eigenvectors so that $\boldsymbol{V}^{-1} = \boldsymbol{V}^\top$. We can now explicitly evaluate (14) to write

$$
1 - \boldsymbol{e}_k^\top \boldsymbol{V} \boldsymbol{\Lambda}^{M+1} \boldsymbol{V}^{-1} \boldsymbol{e}_k = 1 - \left(\frac{(1-K)^{M+1}\rho^{M+1}}{K} + \frac{(K-1)^2 \rho^{M+1}}{(K-1)+(K-1)^2} \right).
$$

Under our assumption that M is odd then $(1-K)^{M+1} = (K-1)^{M+1}$ and this expression simplifies to (11) directly. ■

We note that when $K = 2$, the lower bound on ρ is computed to be 1 for any value of M, hence no admissible choice of ρ will lead to an error probability greater than 0.5 at any stage in the two-user scenario. On the other hand, when $K > 2$ then the lower bound is strictly less than one for all odd values of M and is decreasing in M. The common case of the two-stage LPIC detector ($M = 1$) leads to the following condition on ρ

$$
\frac{1}{\sqrt{K-1}} < \rho \leq 1.
$$

In the limit, as $M \to \infty$ (through all odd values of M) it can be shown that the condition on ρ is

$$
\frac{1}{K-1} < \rho \leq 1.
$$

We note that this condition is equivalent to $\boldsymbol{R}$ having an eigenvalue greater than 2 in the equicorrelated case. The fact that the bound is decreasing in M implies that the performance of the LPIC detector may become worse at later stages when compared to earlier stages, as was also seen in Figure 2.

As a numerical example, consider a communication system with $K = 8$ equipower, equicorrelated users where $a/\sigma = 10$ and $\rho = 0.25$. Computation of (11) under these conditions indicates that $P^{(k)}_{\mathsf{LPIC}}(1) < 0.5$ but $P^{(k)}_{\mathsf{LPIC}}(M) > 0.5$ for all odd values of $M \geq 3$. Figure 3 confirms this analysis. Moreover, note that, at even values of M, the LPIC detector exhibits poor error probability performance with respect to the matched filter detector in this example, yet the error probability does not exceed 0.4 for any even value of M. This example suggests that the error probabilities for odd and even values of M converge to a pair of respective fixed points symmetric around 0.5 as $M \to \infty$. A proof of this phenomenon is developed in the next section.

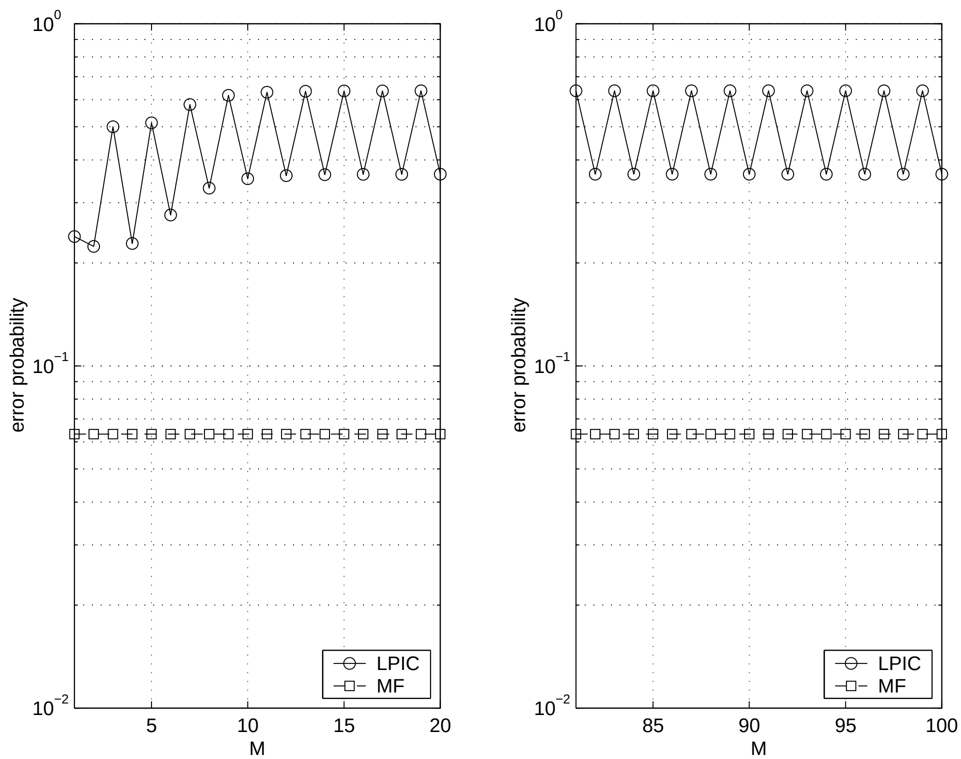

Fig. 3. Example of error probabilities $P^{(k)}_{\mathsf{LPIC}}(M)$ and $P^{(k)}_{\mathsf{MF}}$ for an LPIC detector with M stages of interference cancellation in an equicorrelated, equipower communication system with $K = 8$ users.

V. LPIC Error Probability Divergence as $M \to \infty$

This section analyzes the behavior of the LPIC detector in the asymptotic case where the number of stages M goes to infinity. It has been shown in [8] and [9] that the M-stage LPIC detector converges to the decorrelating detector as $M \to \infty$ when the spectral radius of the crosscorrelation matrix $\rho(\boldsymbol{R})$ is less than 2. This section analyzes the asymptotic behavior of the LPIC detector when $\rho(\boldsymbol{R}) > 2$.

Recall that the M stage LPIC detector may be expressed as

$$\boldsymbol{L}(M) = \sum_{m=0}^{M} (\boldsymbol{I} - \boldsymbol{R})^m.$$

Let $\rho(\boldsymbol{R})$ represent the spectral radius of the crosscorrelation matrix $\boldsymbol{R}$ where $\rho(\boldsymbol{R}) \triangleq \max_k \gamma_k$ and where the $\{\gamma_k\}_{k=1}^K$ is the set of nonnegative eigenvalues of $\boldsymbol{R}$. Note that $\gamma_k = 1 - \lambda_k$ where $\{\lambda_k\}_{k=1}^K$ is the set of eigenvalues of $(\boldsymbol{I} - \boldsymbol{R})$ under the notation established in Section IV. It was shown in [8] and [9] that

$$\boldsymbol{L}(\infty) = \sum_{m=0}^{\infty} (\boldsymbol{I} - \boldsymbol{R})^m = \boldsymbol{R}^{-1}$$

if $\boldsymbol{R}$ is nonsingular and $\rho(\boldsymbol{R}) < 2$. If $\rho(\boldsymbol{R}) > 2$ then there exists at least one λ_k such that $|\lambda_k| > 1$ and it is clear that $\boldsymbol{L}(M)$ does not converge to $\boldsymbol{R}^{-1}$ as $M \to \infty$. The following proposition analyzes the error probability behavior of the LPIC detector when $\rho(\boldsymbol{R}) > 2$ as $M \to \infty$.

Proposition 4: Given $\boldsymbol{R}$ such that $\rho(\boldsymbol{R}) > 2$, let $\{\gamma_\ell\}_{\ell=1}^K$ and $\{\boldsymbol{v}_\ell\}_{\ell=1}^K$ and be the set of nonnegative eigenvalues and associated unit-norm eigenvectors of $\boldsymbol{R}$. Assume that the maximum eigenvalue occurs with algebraic multiplicity p and order the eigenvalues and eigenvectors such that $\gamma_1 = \gamma_2 = \cdots = \gamma_p = \rho(\boldsymbol{R})$ and $\{\boldsymbol{v}_\ell\}_{\ell=1}^p$ are the eigenvectors that constitute a basis for the eigenspace of the maximum eigenvalue. If $\sum_{\ell=1}^p \left(\boldsymbol{e}_k^\top \boldsymbol{v}_\ell\right)^2 \neq 0$ for a fixed desired user k then

$$\lim_{M\to\infty} \alpha_k(2M) = -\lim_{M\to\infty} \alpha_k(2M+1) = \frac{a^{(k)}\sqrt{\rho(\boldsymbol{R}) \sum_{\ell=1}^p \left(\boldsymbol{e}_k^\top \boldsymbol{v}_\ell\right)^2}}{\sigma} \tag{15}$$

and

$$\lim_{M\to\infty} \beta_k(\boldsymbol{d}, 2M) = -\lim_{M\to\infty} \beta_k(\boldsymbol{d}, 2M+1) = \frac{\sqrt{\rho(\boldsymbol{R})} \sum_{\ell=1}^p \boldsymbol{e}_k^\top \boldsymbol{v}_\ell \boldsymbol{v}_\ell^\top \boldsymbol{A}\boldsymbol{d}}{\sigma \sqrt{\sum_{\ell=1}^p \left(\boldsymbol{e}_k^\top \boldsymbol{v}_\ell\right)^2}} \tag{16}$$

where $\alpha_k(M)$ and $\beta_k(\boldsymbol{d}, M)$ are defined in (12). The error probability of the LPIC detector in terms of $\alpha_k(M)$ and $\beta_k(\boldsymbol{d}, M)$ is given in (13).

Proof: Since $\boldsymbol{L}(M)\boldsymbol{R}$ is a real symmetric matrix expressible as a polynomial in $\boldsymbol{R}$, $\boldsymbol{L}(M)\boldsymbol{R}$ is diagonalizable with eigenvectors $\{\boldsymbol{v}_\ell\}_{\ell=1}^K$ and we can write

$$\boldsymbol{L}(M)\boldsymbol{R} = \sum_{\ell=1}^{K} \boldsymbol{v}_\ell \boldsymbol{v}_\ell^\top f(\gamma_\ell, M)$$

where $f(\gamma_\ell, M) = \gamma_\ell \sum_{m=0}^{M} (1-\gamma_\ell)^m$. Similarly,

$$\boldsymbol{L}(M)\boldsymbol{R}\boldsymbol{L}(M) = \sum_{\ell=1}^{K} \boldsymbol{v}_\ell \boldsymbol{v}_\ell^\top g(\gamma_\ell, M)$$

where $g(\gamma_\ell, M) = \gamma_\ell \left(\sum_{m=0}^{M} (1-\gamma_\ell)^m\right)^2$. Under this notation, we can write

$$\begin{aligned}
\alpha_k(M) &= \frac{a^{(k)} \boldsymbol{e}_k^\top \boldsymbol{L}(M)\boldsymbol{R}\boldsymbol{e}_k}{\sigma\sqrt{\boldsymbol{e}_k^\top \boldsymbol{L}(M)\boldsymbol{R}\boldsymbol{L}(M)\boldsymbol{e}_k}} \\
&= \frac{a^{(k)}}{\sigma} \cdot \frac{\sum_{\ell=1}^{p} (\boldsymbol{e}_k^\top \boldsymbol{v}_\ell)^2 f(\rho(\boldsymbol{R}), M) + \sum_{\ell=p+1}^{K} (\boldsymbol{e}_k^\top \boldsymbol{v}_\ell)^2 f(\gamma_\ell, M)}{\sqrt{\sum_{\ell=1}^{p} (\boldsymbol{e}_k^\top \boldsymbol{v}_\ell)^2 g(\rho(\boldsymbol{R}), M) + \sum_{\ell=p+1}^{K} (\boldsymbol{e}_k^\top \boldsymbol{v}_\ell)^2 g(\gamma_\ell, M)}}.
\end{aligned}$$

Inspection of f and g and the fact that $\sum_{\ell=1}^{p} \left(\boldsymbol{e}_k^\top \boldsymbol{v}_\ell\right)^2 > 0$ implies that the summations over $\ell = 1, \ldots, p$ in the numerator and denominator grow without bound as $M \to \infty$ for $\rho(\boldsymbol{R}) > 2$. Although the summations over $\ell = p+1, \ldots, K$ may also grow without bound as $M \to \infty$, they do not grow as fast as the summations over $\ell = 1, \ldots, p$, hence

$$\begin{aligned}
\lim_{M\to\infty} \alpha_k(M) &= \frac{a^{(k)}}{\sigma} \lim_{M\to\infty} \frac{\sum_{\ell=1}^{p} (\boldsymbol{e}_k^\top \boldsymbol{v}_\ell)^2 f(\rho(\boldsymbol{R}), M)}{\sqrt{\sum_{\ell=1}^{p} (\boldsymbol{e}_k^\top \boldsymbol{v}_\ell)^2 g(\rho(\boldsymbol{R}), M)}} \\
&= \frac{a^{(k)} \sqrt{\sum_{\ell=1}^{p} (\boldsymbol{e}_k^\top \boldsymbol{v}_\ell)^2}}{\sigma} \lim_{M\to\infty} \frac{f(\rho(\boldsymbol{R}), M)}{\sqrt{g(\rho(\boldsymbol{R}), M)}}.
\end{aligned}$$

Since

$$\begin{aligned}
\frac{f(\rho(\boldsymbol{R}), M)}{\sqrt{g(\rho(\boldsymbol{R}), M)}} &= \sqrt{\rho(\boldsymbol{R})} \cdot \operatorname{sgn}\left(\sum_{m=0}^{M} (1-\rho(\boldsymbol{R}))^m\right) \\
&= \begin{cases} \sqrt{\rho(\boldsymbol{R})} & \text{for even values of M} \\ -\sqrt{\rho(\boldsymbol{R})} & \text{for odd values of M} \end{cases}
\end{aligned}$$

then (15) follows directly. The proof of (16) follows similarly. ■

One implication of this result is given in the following Corollary.

Corollary 1: Under the assumptions of Proposition 4, $P^{(k)}_{\mathsf{LPIC}}(2M)$ and $P^{(k)}_{\mathsf{LPIC}}(2M+1)$ converge to a pair of respective fixed points symmetric about 0.5 as $M \to \infty$.

Proof: This can be seen from the fact that

$$\lim_{M\to\infty} \left[Q\left(\alpha_k(M) + \beta_k(\boldsymbol{d}, M)\right) + Q\left(\alpha_k(M+1) + \beta_k(\boldsymbol{d}, M+1)\right)\right] = Q(x) + Q(-x) = 1$$

hence, from (13), we can write

$$\lim_{M\to\infty} \left[P^{(k)}_{\mathsf{LPIC}}(M) + P^{(k)}_{\mathsf{LPIC}}(M+1)\right] = \frac{1}{2^{K-1}} \sum_{\boldsymbol{d}\in\mathcal{D}_k} 1 = 1$$

■

We note that, under the "divergence conditions" of Proposition 4, Corollary 1 implies that the error probability of the LPIC detector will oscillate around the value 0.5 for large values of M. Moreover, since $\lim_{M\to\infty} \alpha(2M+1) < 0$, the error probability of the LPIC detector will be greater than 0.5 for large odd values of M. The numerical example of Figure 3 demonstrates this behavior. This behavior is in contrast to the asymptotic behavior of the LPIC detector when $\rho(\boldsymbol{R}) < 2$ where the error probability converges to a single fixed point equal to the decorrelator's error probability.

In the equicorrelated case where $\boldsymbol{R}_{ij} = \rho$ for all $i \neq j$, we showed in Proposition 3 that the normalized eigenvector associated with the unique maximum eigenvalue is given by $\boldsymbol{v}_1 = [1, 1, \ldots, 1]^\top / \sqrt{K}$. In this case, it is clear that $\left(\boldsymbol{e}_k^\top \boldsymbol{v}_1\right)^2 = 1/K > 0$ for any user k. This implies that all users in the equicorrelated system will exhibit divergent error probabilities in the sense of Proposition 4 and Corollary 1 when $\rho(\boldsymbol{R}) > 2$.

For general signature crosscorrelation matrices, no eigenvector of $\boldsymbol{R}$ can have elements all equal to zero, hence it is impossible to satisfy the condition $\sum_{\ell=1}^{p} \left(\boldsymbol{e}_k^\top \boldsymbol{v}_\ell\right)^2 = 0$ for all $k \in \{1, \ldots, K\}$. This implies that there will always exist at least one user whose error probability will diverge when $\rho(\boldsymbol{R}) > 2$. It is tempting to think that if $\rho(\boldsymbol{R}) > 2$, all users must exhibit divergent error probabilities but the following example indicates otherwise.

Suppose we have $K = 5$ users and a crosscorrelation matrix $\boldsymbol{R}$ given by

$$\boldsymbol{R} = \frac{1}{11}\begin{bmatrix} 11 & -1 & -1 & -1 & 3 \\ -1 & 11 & 7 & 7 & 7 \\ -1 & 7 & 11 & 7 & 7 \\ -1 & 7 & 7 & 11 & 7 \\ 3 & 7 & 7 & 7 & 11 \end{bmatrix}. \tag{17}$$

The spectral radius of $\boldsymbol{R}$ is given by its largest eigenvalue which is computed to be $\rho(\boldsymbol{R}) = 32/11 = \gamma_1 > 2$ and it can be verified that all other eigenvalues of $\boldsymbol{R}$ are in the open interval $(0, 2)$. The unit-norm eigenvector associated with the maximum eigenvalue is given as

$$\boldsymbol{v}_1 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \end{bmatrix}^\top.$$

It is clear that setting $k = 1$ in this case yields $\left(\boldsymbol{e}_k^\top \boldsymbol{v}_1\right)^2 = 0$. We can compute that $\lim_{M\to\infty} \alpha_1(M) \approx 0.8528a^{(k)}/\sigma$, that $\lim_{M\to\infty} \beta_1(\boldsymbol{d}, M) = 0$, and that the error probability of user 1 in this case converges to that of the decorrelating detector. The key to this example is that the eigenvector associated with the maximum eigenvalue has a zero in a fortuitous location for user 1. The error probability of each other user $k \neq 1$ diverges in the sense of Proposition 4 and Corollary 1. This example confirms our claim that not every user will necessarily exhibit error probability divergence when $\rho(\boldsymbol{R}) > 2$ since there is no guarantee that the eigenvector associated with the maximum eigenvalue has all nonzero entries for general $\boldsymbol{R}$.

The natural question to ask is when does $\boldsymbol{R}$ have an eigenvector with nonzero entries associated with an eigenvalue greater than two? We have not been able to classify all such crosscorrelation matrices, but Perron's Theorem [22, pg. 500] and its extensions identify a large class of such matrices. The theorem states that

Theorem 1: Perron's Theorem. If $\boldsymbol{A}$ is an $n \times n$ matrix with positive entries, then

1. $\rho(\boldsymbol{A}) > 0$ and is a simple (multiplicity one) eigenvalue of $\boldsymbol{A}$.
2. The eigenvector associated with $\lambda = \rho(\boldsymbol{A})$ has positive entries.

Although Perron's Theorem may be generalized from the class of all positive matrices to particular classes of nonnegative matrices [22, pp. 508, 516], it does not extend to the case of signature crosscorrelation matrices with negative elements, e.g. (17). On the other hand, the implications of Perron's Theorem are stronger than necessary and simulations suggest that it is actually fairly difficult to find valid signature crosscorrelation matrices with an eigenvalue greater than two and an associated eigenspace with one or more null dimensions. In Section VI we reinforce this intuition by showing via simulations that this event occurs with low probability in the large-system case with random spreading sequences.

One additional caveat with respect to Corollary 1 is necessary. Figure 4 plots the error probability of user 1 versus M for a 4 user, equipower communication system with the signature crosscorrelation matrix

$$\boldsymbol{R} = \frac{1}{5}\begin{bmatrix} 5 & -1 & -2 & -2 \\ -1 & 5 & -2 & 4 \\ -2 & -2 & 5 & -1 \\ -2 & 4 & -1 & 5 \end{bmatrix}$$

In this case, $\rho(\boldsymbol{R}) \approx 2.0485$ and none of the elements of the eigenvector associated with the unique maximum eigenvalue are equal to zero. Corollary 1 indicates that the error probability of the LPIC detector will converge to a pair of fixed points centered around 0.5 for each user in this system and Figure 4 confirms that this is indeed the case for user 1. Nevertheless, we note that there are several values of M for which the M-stage LPIC detector exhibits an error probability several orders of magnitude better than the matched filter detector. This example shows that, even in cases when the LPIC detector is known to diverge, there may exist values of $M < \infty$ for which the M-stage LPIC detector performs quite well. Unfortunately, there does not appear to a closed form expression for $\arg\min_M P^{(k)}_{\mathsf{LPIC}}(M)$ and the matter of finding a simple indicator as to when the performance of the LPIC detector will deteriorate or improve with the application of additional stages remains an open problem.

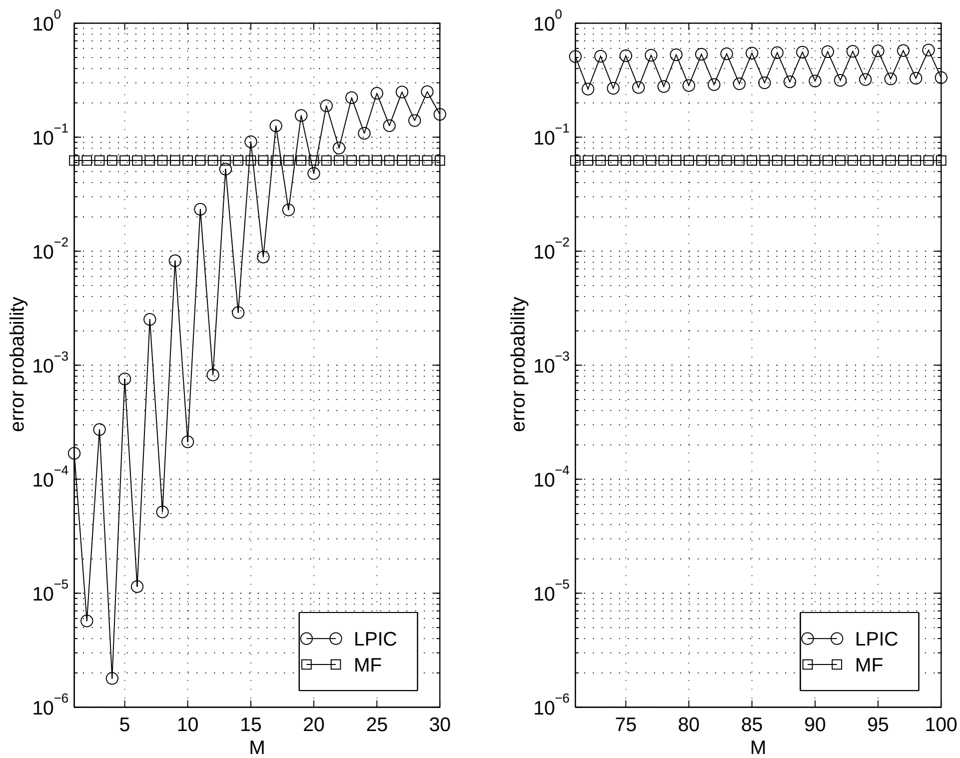

Fig. 4. Example of error probabilities $P^{(1)}_{\mathsf{LPIC}}(M)$ and $P^{(1)}_{\mathsf{MF}}$.

VI. Random Signature Sequences: Large-System Analysis

This section considers the implications of the general results developed in the prior sections to a CDMA communication system where the users' spreading sequences are chosen randomly. Specifically, we denote the n^{th} element of the k^{th} user's length-N spreading sequence as $s_n^{(k)}$ where $s_n^{(k)} \in \{\pm 1\}$ and $P(s_n^{(k)} = -1) = P(s_n^{(k)} = +1) = 1/2$ for all $k \in \{1, \ldots, K\}$ and $n \in \{1, \ldots, N\}$. We also assume that the elements of the spreading sequences are chosen independently such that $\mathrm{E}[s_n^{(k)} s_{n'}^{(k')}] = 0$ unless $k = k'$ and $n = n'$. In this case, we can express the signature crosscorrelation matrix as

$$\boldsymbol{R} = \frac{1}{N} \boldsymbol{S}^\top \boldsymbol{S} \tag{18}$$

where $\boldsymbol{S}_{nk} = s_n^{(k)}$ is spreading sequence matrix constructed such that the k^{th} column, denoted by $\boldsymbol{s}_k$, is equal to the k^{th} user's spreading sequence. To obtain analytical results, we focus on the "large-system" scenario (described in [23] and [24]), where the spreading gain N and the number of users K both approach infinity but their ratio $\beta = K/N$ converges to a fixed constant.

We note that the results of this section apply both to the case where the users spreading sequences are initially chosen randomly but remain fixed over the duration of their transmission and to the case where a new set of random spreading sequences are generated at each bit interval. In each case, we analyze the expected performance of each user averaged over all possible realizations of the noise, transmitted data, and spreading sequences.

A. LPIC vs. HPIC Performance Comparison

Since Proposition 1 holds for arbitrary $\boldsymbol{R}$ then it also holds for $\boldsymbol{R}$ described by (18). It turns out that the large-system random spreading sequences case allows us to reconsider Proposition 1 without the use of the Gaussian approximation to achieve an *exact* comparison of the MSE of the interference estimates for the two-stage LPIC and HPIC detectors.

In the large-system case it was shown in [1, pp. 116] that the average error probability for the matched filter detector with random spreading sequences can be written without approximation as

$$\mathrm{E}[P(\mathrm{sgn}(y^{(\ell)}) \neq b^{(\ell)})] = \mathrm{E}[P_{\mathsf{MF}}^{(\ell)}] = Q\left(\frac{a^{(\ell)}}{\sqrt{\sigma^2 + \beta \bar{a}^2}}\right)$$

where

$$\bar{a}^2 = \lim_{K\to\infty} \frac{1}{K} \sum_{k\neq\ell} (a^{(k)})^2.$$

This result in combination with (4) allows us to express the average interference estimate MSE for the ℓ^{th} user of the two-stage HPIC detector as

$$\mathrm{E}[\mathrm{MSE}_{\mathsf{HPIC}}^{(\ell)}] = 4(a^{(\ell)})^2 Q\left(\frac{a^{(\ell)}}{\sqrt{\sigma^2 + \beta \bar{a}^2}}\right).$$

It is also possible to calculate the average interference estimate MSE of the ℓ^{th} user of the two-stage LPIC detector in the large-system random spreading sequences case without approximation as

$$\begin{aligned}
\mathrm{E}[\mathrm{MSE}_{\mathsf{LPIC}}^{(\ell)}] &= \mathrm{E}\left[(e_{\mathsf{LPIC}}^{(\ell)})^2 \mid b^{(\ell)}\right] \\
&= \mathrm{E}\left[\left(\sum_{k\neq\ell} \boldsymbol{s}_\ell^\top \boldsymbol{s}_k a^{(k)} b^{(k)} + \sigma n^{(\ell)}\right)^2\right] \\
&= \sigma^2 + \beta \lim_{K\to\infty} \frac{1}{K} \sum_{k\neq\ell} (a^{(k)})^2 \\
&= \sigma^2 + \beta \bar{a}^2
\end{aligned}$$

where we have used the property that

$$\mathrm{E}[\boldsymbol{s}_\ell^\top \boldsymbol{s}_k \boldsymbol{s}_\ell^\top \boldsymbol{s}_j] = \begin{cases} 1/N & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}.$$

Hence, in the large-system random spreading sequences case, we can write the *exact* expression

$$\mathrm{E}[\mathrm{MSE}_{\mathsf{HPIC}}^{(\ell)}] = 4(a^{(\ell)})^2 Q\left(\frac{a^{(\ell)}}{\sqrt{\mathrm{E}[\mathrm{MSE}_{\mathsf{LPIC}}^{(\ell)}]}}\right)$$

which leads to the following proposition.

Proposition 5: For random $\boldsymbol{R}$ given by (18), arbitrary fixed σ, $\boldsymbol{A}$, and ℓ, $\mathrm{E}[\mathrm{MSE}^{(\ell)}_{\mathsf{LPIC}}] > \mathrm{E}[\mathrm{MSE}^{(\ell)}_{\mathsf{HPIC}}]$ asymptotically as $K \to \infty$, $N \to \infty$, and $K/N \to \beta$.

Proof: The proof follows that of Proposition 1. ■

If we follow the intuition that better interference estimates should lead to better error probability performance then this proposition suggests that the two-stage HPIC detector is uniformly superior to the two-stage LPIC detector in terms of error probability for a large CDMA system with synchronous users and random spreading sequences.

B. Comparison to the Matched Filter Detector

Direct interpretation of Proposition 2 in the large-system case with random spreading sequences is difficult since, unlike the matched filter, an exact expression for the LPIC detector's average probability of error $\mathrm{E}[P^{(\ell)}_{\mathsf{LPIC}}(M)]$ is difficult to obtain even for the two-stage case. Rather than directly comparing the error probabilities of the MF and LPIC detectors, we can instead compare their output signal-to-interference-plus-noise ratios (SINR).

In the random spreading sequence scenario, SINR is defined [1, pp. 280] as the ratio of the second moments of the desired component to the interference (background noise plus multiaccess interference) component averaged with respect to transmitted data, noise, and random spreading sequences. One justification for analyzing the output SINR of a multiuser detector is that certain multiuser detectors exhibit a soft decision statistic that may be described without approximation as a Gaussian random variable in the large-system, random spreading sequence scenario. In this case, the multiuser detector's error probability and output SINR are related by the expression

$$\mathrm{E}[P^{(\ell)}_{\mathsf{MUD}}] = Q\left(\sqrt{\mathrm{SINR}^{(\ell)}_{\mathsf{MUD}}}\right).$$

We note that this property holds for the MF detector but the same is not immediately true for the M-stage LPIC detector. Although numerical evidence suggests that decision statistics of the M-stage LPIC detector may indeed be Gaussian, a proof of this property appears to be difficult and remains an open problem. Nevertheless, we analyze the SINR of the LPIC detector in this section under the premise that SINR and error probability are often closely related even in the case when the decision statistics are not exactly Gaussian. Moreover we note that SINR is also an appropriate performance measure if the

LPIC detector's outputs are to be used by a soft decision channel decoder.

Proposition 6: Assume the large-system scenario with randomly chosen spreading sequences. Let $\mathrm{SINR}_{\mathsf{MF}}^{(\ell)}$ and $\mathrm{SINR}_{\mathsf{LPIC}}^{(\ell)}(1)$ denote the SINR of the ℓ^{th} user at the output of the MF and the two-stage LPIC detectors respectively. Then $\mathrm{SINR}_{\mathsf{MF}}^{(\ell)} > \mathrm{SINR}_{\mathsf{LPIC}}^{(\ell)}(1)$ if and only if $\beta > \frac{1}{3} - \frac{1}{3\bar{a}^2/\sigma^2}$ where $\bar{a}$ is given in (20).

Proof: The asymptotic SINR of the matched filter detector for the large-system random spreading sequence scenario is given in [1, pp. 281] as

$$\mathrm{SINR}_{\mathsf{MF}}^{(\ell)} = \frac{(a^{(\ell)})^2}{\sigma^2 + \beta\bar{a}^2} \tag{19}$$

where

$$\bar{a}^2 = \lim_{K\to\infty} \frac{1}{K} \sum_{k\neq\ell} (a^{(k)})^2. \tag{20}$$

The asymptotic SINR of the approximate decorrelator (equivalent to the two-stage LPIC detector) for a large CDMA system with random spreading sequences is given in [11] as

$$\mathrm{SINR}_{\mathsf{AD}}^{(\ell)} = \mathrm{SINR}_{\mathsf{LPIC}}^{(\ell)}(1) = \frac{(a^{(\ell)}(1-\beta))^2}{\sigma^2(1-\beta+\beta^2) + \bar{a}^2(\beta^2+\beta^3)}$$

Comparison of $\mathrm{SINR}_{\mathsf{LPIC}}^{(\ell)}(1)$ to $\mathrm{SINR}_{\mathsf{MF}}^{(\ell)}$ reveals that $(a^{(\ell)})^2$ can be factored out of both expressions and that there is no amplitude threshold behavior as seen in Proposition 2. Instead, the relationship between $\mathrm{SINR}_{\mathsf{LPIC}}^{(\ell)}(1)$ and $\mathrm{SINR}_{\mathsf{MF}}^{(\ell)}$ depends on β and the ratio of mean interference to noise power, $\bar{a}^2/\sigma^2$. Setting the ratio

$$\frac{\mathrm{SINR}_{\mathsf{MF}}^{(\ell)}}{\mathrm{SINR}_{\mathsf{LPIC}}^{(\ell)}(1)} > 1$$

and solving for β, we get

$$\beta > \frac{1}{3} - \frac{1}{3\bar{a}^2/\sigma^2}. \tag{21}$$

■

We can draw two conclusions from (21). First, in the case when $\bar{a}^2/\sigma^2$ is small, the MF detector intuitively performs better than the two-stage LPIC detector because the multiple access interference estimates, upon which the LPIC detector crucially relies, are unreliable in this region. In fact, (21) implies that the MF detector outperforms (in terms of asymptotic SINR) the two-stage LPIC detector for any $\beta > 0$ if $\bar{a}^2/\sigma^2 \leq 1$. Second, for any value of $\bar{a}^2/\sigma^2$, (21) implies that the MF detector outperforms the two-stage LPIC detector if $\beta > 1/3$. In this operating region, the ratio of the number of users to the spreading gain is large enough such that the LPIC detector's multiple access interference estimates are unreliable and interference cancellation is detrimental to performance. This behavior was first observed in [11] in the noise free case.

An extension to Proposition 6 for the general case of the M-stage LPIC detector remains an open problem. Calculation of the asymptotic SINR for an M-stage LPIC detector for arbitrary $M > 1$ is more complicated and appears to require computation of the moments of the random eigenvalues of $\boldsymbol{R}$ from the distribution given in [23]. An analytical comparison of the MF and M-stage LPIC detector asymptotic SINRs remains an open problem.

C. LPIC Error Probability Divergence and Asymptotic Results

Proposition 3 does not have direct application in the case when the crosscorrelation matrix $\boldsymbol{R}$ is random since it involves selection of some particular $\boldsymbol{R}$ to show error probability divergence of the LPIC detector. However, Proposition 4 and Corollary 1 do have a meaningful interpretation due to the following theorem by Bai and Yin [25].

Theorem 2: (Bai and Yin). Let $\boldsymbol{S}$ be a $N \times K$ matrix of independent and identically distributed random variables with zero mean and unit variance. Let $\boldsymbol{R} = \frac{1}{N}\boldsymbol{S}^\top \boldsymbol{S}$. If $\mathrm{E}[|\boldsymbol{S}_{11}|^4] < \infty$, then, as $K \to \infty, N \to \infty, \frac{K}{N} \to \beta \in (0,1)$, the largest eigenvalue of $\boldsymbol{R}$ converges to $(1+\sqrt{\beta})^2$ with probability one. The minimum eigenvalue converges to $(1-\sqrt{\beta})^2$ with probability one.

We apply Theorem 2 in the following proposition.

Proposition 7: Assume the large-system scenario with randomly chosen spreading sequences. If $(\sqrt{2}-1)^2 < \beta < 1$ then $\rho(\boldsymbol{R}) > 2$ with probability one and $P^{(k)}_{\mathsf{LPIC}}(M)$ diverges in the sense of Proposition 4 and Corollary 1 as $M \to \infty$ for at least one user.

Proof: Theorem 2 indicates that the largest eigenvalue of $\boldsymbol{R}$ converges to a deterministic value in a large-system, random spreading sequence scenario. Since $\boldsymbol{R}$ is nonnegative definite, the maximum eigenvalue is equivalent to the spectral radius $\rho(\boldsymbol{R})$, hence

$$\rho(\boldsymbol{R}) \xrightarrow[\text{a.s.}]{} (1+\sqrt{\beta})^2. \tag{22}$$

Manipulation of (22) yields

$$\rho(\boldsymbol{R}) > 2 \iff \beta > \left(\sqrt{2}-1\right)^2 \approx 0.17$$

with probability one. Proposition 4 indicates that, when $\rho(\boldsymbol{R}) > 2$, each user k that does not satisfy the property $\sum_{\ell=1}^{p}(\boldsymbol{e}_k^\top \boldsymbol{v}_\ell)^2 = 0$ will exhibit the divergent asymptotic error probabilities described by (15) and (16) as $M \to \infty$. Since the eigenvectors $\{\boldsymbol{v}_\ell\}_{\ell=1}^{p}$ cannot have all elements equal to zero, error probability divergence in the sense of Proposition 4 and Corollary 1 occurs with probability one for at least one user when $(\sqrt{2}-1)^2 < \beta < 1$.

■

We note that, since all users do not necessarily exhibit error probability divergence for a particular realization of $\boldsymbol{R}$, different realizations of $\boldsymbol{R}$ may cause different users to exhibit error probability divergence in the sense of Proposition 4 and Corollary 1. Hence, in the case when the users' spreading sequences change between bit intervals, Proposition 7 does not imply that the *average* error probability (over all possible realizations of $\boldsymbol{R}$) necessarily diverges for any of the users. However, simulation evidence suggests that the eigenvectors forming the basis for the eigenspace of the largest eigenvalue of the random signature crosscorrelation matrix $\boldsymbol{R}$ have elements equal to zero with very low probability for sufficiently large K and N. This then implies that, for nearly all realizations of $\boldsymbol{R}$, all users will exhibit error probability divergence in the sense of Proposition 4 and Corollary 1 as $M \to \infty$. Figure 5 confirms this behavior by examining the average error probability performance of the LPIC detector in the case when $\boldsymbol{R}$ changes between bit intervals.

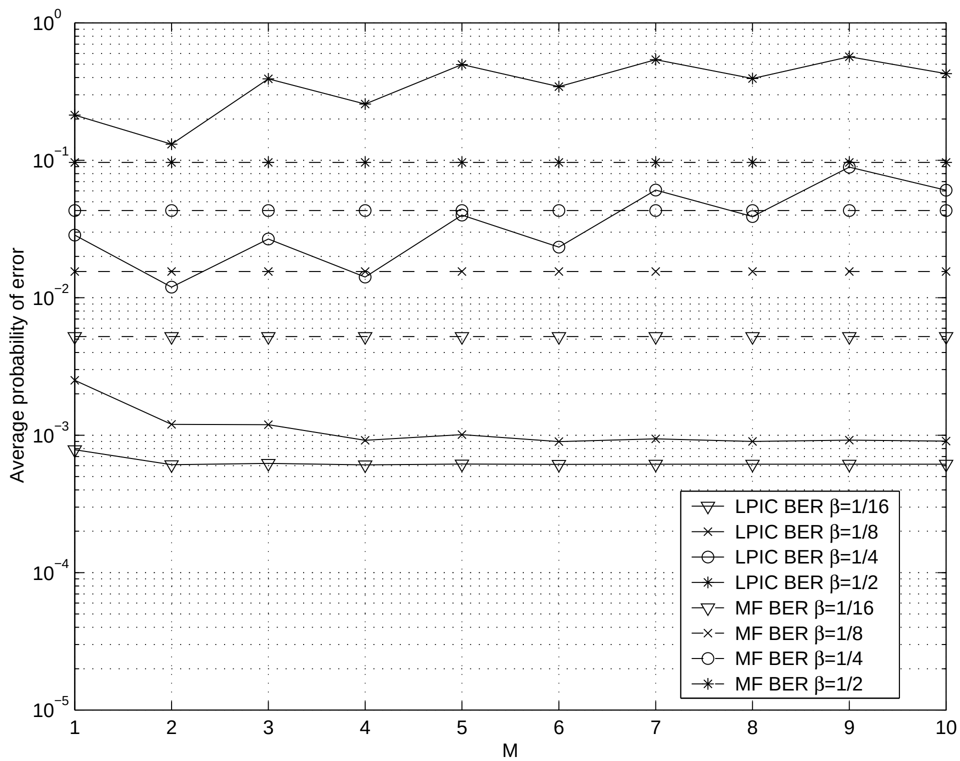

Fig. 5. Expected error probability versus number of LPIC interference cancellation stages for a CDMA system with spreading gain $N = 256$, random spreading sequences, and $a^{(k)}/\sigma = 1/0.3$ for all users. The results shown are averaged over 10^5 realizations of the random spreading sequences, transmitted bits, and noise.

VII. Conclusions

This paper examined several performance aspects of the multistage LPIC detector. We presented analytical evidence that supports the recent simulation evidence of other authors suggesting that Varanasi and Aazhang's HPIC detector may outperform the LPIC detector in many common operating scenarios. We derived a closed form expression for a sufficient threshold which, if exceeded by the desired user's amplitude, causes the matched filter detector to outperform the LPIC detector in terms of error probability for the desired user. We developed an explicit description of a set of signature crosscorrelation matrices, parameterized by the number of interference cancellation stages M, such that the the LPIC detector exhibits an error probability greater than 0.5 for binary signaling. The behavior of the LPIC detector was also investigated in the asymptotic case when $M \to \infty$. Under conditions such that the LPIC detector does not converge to the decorrelator, we derived a closed form expression for the asymptotic error probability of the LPIC detector and showed that it converges to a pair of fixed points centered around 0.5.

The implications of the prior results were studied for CDMA communication systems with large bandwidth, a large number of users, and random spreading sequences. We showed that the HPIC detector is uniformly superior to the LPIC detector in terms of interference estimator performance in this scenario. We also showed that the two-stage LPIC detector exhibits worse asymptotic output SINR performance than the MF detector when $K/N > 1/3$ for any choice of desired user amplitude and interference or noise powers. Applying a recent result from random matrix theory, we showed the asymptotic result that the M-stage LPIC detector will not converge to the decorrelating detector and that at least one user will exhibit an error probability that converges to a pair of fixed points centered around 0.5 as $M \to \infty$ if $0.17 < K/N < 1$.

The results presented in this paper are intended to advance our understanding of PIC detector since they offer many attractive features in terms of computational complexity and decision latency. Indeed, there are many documented cases of good performance by both the LPIC and HPIC detectors in the literature. However, the results derived in this paper are intended to fill in some of the gaps in our understanding of PIC detectors and to serve as cautionary guidelines as to when the LPIC detector may exhibit undesirable performance.

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