Steady State Kalman Filter Behavior for Unstabilizable Systems

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***Abstract*— Some important textbooks on Kalman Filters suggest that positive semidefinite solutions to the filtering Algebraic Riccati Equation (ARE) cannot be stabilizing should the underlying state variable realization be unstabilizable. We show that this is false. Questions of uniqueness of positive semidefinite solutions of the ARE are also unresolved in the absence of stabilizability. Yet fundamental performance issues in modern communications systems hinge on Kalman Filter performance absent stabilizability. In this paper we provide a positive semidefinite solution to the ARE for detectable systems that are not stabilizabile and show that it is unique if the only unreachable modes are on the unit circle.**

***Index Terms*— Riccati Equation, Kalman Filter, Stability, Uniqueness.**

I. INTRODUCTION

This paper concerns the steady state behavior of the Kalman Filter for the system described by the discrete time state variable realization

$$x[k+1] = Fx[k] + Gw[k] \quad (1)$$
$$y[k] = Hx[k] + v[k], \quad (2)$$

where $F \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times q}$, $H \in \mathbb{R}^{p\times n}$, and $x[0] \sim N(\bar{x}_0, P_0)$ and the white Gaussian random processes $w[k] \sim N(0, Q)$ and $v[k] \sim N(0, R)$ are mutually uncorrelated.

Even though the Kalman Filter is itself time varying, its asymptotic behavior is gauged by its steady state version. Thus for example, the steady state one-step-ahead error covariance matrix P that satisfies the Algebraic Riccati Equation (ARE), [1]

$$P = F\left(P - PH^\top \left(HPH^\top + R\right)^{-1} HP\right) F^\top + GQG^\top \quad (3)$$

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provides asymptotic performance bounds. Furthermore, a central question relates to steady state filter stability. Specifically with K the steady state Kalman Gain obeying,

$$K = FPH^\top \left(HPH^\top + R\right)^{-1} \quad (4)$$

this is related to whether or not $F - KH$ is Schur, i.e. has all eigenvalues in the open unit disc. This question and the existence and uniqueness of a positive semidefinite solution to (3) have been extensively studied, [1]-[4], sometimes through the dual Riccati equation featuring in Linear Quadratic Regulator (LQR) theory, [5]. In the sequel a positive semidefinite solution to (3) will be called *stabilizing* if under (4) $F - KH$ is Schur.

Most studies, see e.g. [1], assume that $[F, H]$ is detectable and $[F, GQ^{1/2}]$ is *stabilizable*. Under these conditions it is known that the ARE has a uinque positive semidefinite stabilizing solution. The detectability condition is cleary necessary. In fact without it the Kalman Filter is not even meaningful as unstable modes are just not observable from the measurements.

This paper revisits this generations old question, by asking if the stabilizability condition is indeed necessary for the steady state Kalman filter to be stable? For that matter does its relaxation necessarily render the solution(s) of (3) meaningless? After all mere detectability ensures the existence of an observer gain L such that $F - LH$ is Schur, thereby ensuring the existence of a stable linear observer, [3]. And Kalman filter is the linear minimum variance filter, even without the Gaussian assumption, [1].

We note that [15] addreses the Kalman Filter, as opposed to the steady state Kalman Filter, behavior when the stabilizability assumption is violated. It shows that lack of stabilizability may preclude the noise free Kalman filter, from being exponentially stable. As described in Section II, it proceeds to provide sufficient conditions under which the Kalman filter, again in the noise free case, is asymptotically stable.

There are some misconceptions regarding the behavior of the steady state Kalman filter, absent stabilizabilty. The following, line in an influential textbook, is illustrative: "A more complicated argument can be used to conclude that if P exists with $F - KH$ asymptotically stable, one must have complete stabilizability of $[F, GQ^{1/2}]$." This same textbook has many exercises that implicitly assume this statement to be true. It is also noteworthy that, MATLAB dismisses requests to solve the control ARE arising in LQR theory, by providing an error message in the absence of detectability. For the filtering ARE of interest here, stabilizability has the same role as detectability in the control ARE.

Beyond the intellectual curiosity of answering a question that remains open after the five plus decades that have elapsed since the Kalman filter was derived, this question has implications that go beyond just grappling with a pathology. Consider in particular, Distributed Multiple Input Multiput Output (DMIMO) communications. Multiple Input Multiput Output (MIMO) communications, refering to receiver and transmitters with multiple antennae, promise to revolutionize contemporary and emerging communications systems, [6], [7].

The theory and practice of MIMO communication has matured to the point where MIMO is now an integral component of several recent WiFi and cellular standards, such as 802.11n, 802.11ac, long-term evolution (LTE), WiMAX, and International Mobile Telecommunications (IMT)-Advanced. While the advantages of MIMO are significant, the applicability of MIMO is often limited by physical and economic constraints. For example, the form factor of handheld devices typically limits the number of antennas to only one or two. Even for infrastructure nodes such as access points and base stations, multiple antennas may be too bulky at large carrier wavelengths, e.g., “white space” frequencies with carrier wavelengths as large as 6 meters.

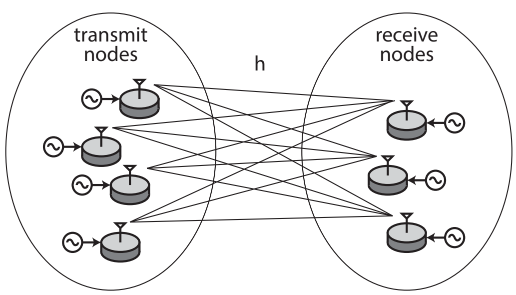

Fig. 1. Distributed MIMO system model with N_t transmit nodes and N_r receive nodes. Each node possesses a single antenna and an independent oscillator.

To overcome these limitations, information theorists proposed more than three decades ago, the concept of DMIMO, where transmitters and receivers cooperatively form virtual antenna arrays, [8]. The basic concept is depicted in Figure 1. The key practical impediment to realizing this concept is the fact, that unlike a centralized array system, in a virtual array, as also depicted in this figure, each node operates its own oscillator. Coherent communications require that these oscillators, whose frequency and phase undergo Brownian motion drift, be tightly synchronized, [10]. More precisely, the unwrapped phase of these oscillators can be modeled as the output of a double integrator excited in each dimension by white noise, [13], [14]. As a result these oscillators quickly, in a matter of milliseconds, drift out of synchrony, [9], [10]. In the long term, they can be modelled as

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + w(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + v(t).$$

It turns out, that in time spans over which synchronization is lost the second element of $w(t)$ has too small a variance to have a significant impact. Theoretical undestanding of fundamental performance bounds, thus calls for its variance to be treated as being zero.

If one treats this second element to be zero, the resulting discrete time system over sampling intervals of T, is as in (1), (2) with

$$F = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, G = \begin{bmatrix} T \\ 0 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \end{bmatrix}. \qquad (5)$$

This is not a stabilizable system. The synchronization problem has been addressed, successfully by us, [11], [9], [10] by using a Kalman Filter to track the states of the various oscillators. The resulting performance bounds, and indeed how frequently one must resynchronize, directly relate to the questions we pose here.

Since the submission of this paper, we have recently discovered an excellent paper by Chan et. al.,[12] that removes the misconceptions manifest in the quoted statement, and shows that positive semidefinite solutions of the ARE do exist despite lack of stabilizability, and may even be stabilizing. In light of the results of [12], we focus here on the following things: (A) In the absence of stabilizability Chan et. al. show that a positive semidefinite solution to the ARE exists, though the solution may not be positive definite. They do not show how to construct this solution. We provide a construction. This is particularly important as MATLAB cannot provide a solution. (B) Since our interest is primrily on settings of [11] where the unstabilizable modes are on the unit circle, we focus on that case and resolve an uniqueness issue left open in [12]. We show that if all unreachable modes are in the closed unit disc then the positive semidefinite solution is unique. (C) We consider a special case with a totally unstable F and zero process noise that yields a very interesting solution to the ARE.

A first order analysis is in Section III. Section IV provides the construction of a postive semidefinite solution of the ARE under detectability. Section V considers a special case where the process noise is zero and F^{-1} is Schur. It is shown that at least one positive definite solution generates an $F-KH$ that is similar to $F^{-\top}$, directly generalizing the scalar example in an appealing way. Section VI proves that the solution in Section IV is unique when the unstabilizable modes are on or inside the unit circle. Section II provides prelminaries and Section VII the conclusion.

II. Preliminaries

This Section provides some preliminary results including recounting the main results of [15] and [12]. First observe that the one step ahead optimal state estimate $\hat{x}$ provided by Kalman Filter equations are: With $\hat{x}[0]=\bar{x}_0$ and $P[0]=P_0$,

$$\hat{x}[k+1]=(F-K[k]H)\hat{x}[k]+K[k](y[k]-H\hat{x}[k]) \quad (6)$$

$$K[k]=FP[k]H^\top\left(HP[k]H^\top+R\right)^{-1} \quad (7)$$

$$P[k+1] = FP[k]F^\top+GQG^\top-FP[k]H^\top (HP[k]H^\top+R)^{-1}HP[k]F^\top. \quad (8)$$

By the stability of the Kalman Filter we refer to the stability of

$$z[k+1]=(F-K[k]H)z[k]. \quad (9)$$

The following is a standing assumption of this paper:

Assumption 1: The pair $[F,H]$ is completely detectable and $R=R^\top>0$..

In addition to Assumption 1, if the pair $[F,GQ^{1/2}]$ is completely stabilizable, then (9) is exponentially asymptotically stable (eas), [15]. Lack of stabilizability in general precludes (9) from being eas. However, [15] provides milder conditions for asymptotic stability through the following theorem.

Theorem 1: Consider (1), (2) under (7), (8) and Assumption 1. Define:

$$W[k]=F^kP_0F^{\top k}+\sum_{i=1}^{k}F^{k-i}GQG^\top F^{\top(k-i)}. \quad (10)$$

Then (9) is asymptotically stable if there exists a $k\geq 0$ such that $W[k]$ is nonsingular.

Thus a positive definite P_0 for instance, suffices for the asymptotic stability of (9). The positive definite nature of the summation in (10) is equivalent to $[F,GQ^{1/2}]$ being completely reachable.

Finally we recount the main pertinent result of [12], which refers to a so called *strong solution* of the ARE: Specifically a positive semidefinte $P=P^\top$ that satisfies (3) is called a strong solution of the ARE if under (4) $F-KH$ has all eigenvalues in the closed unit disc. Observe that a strong solution is not necessarily a stabilizing solution. Then [12] proves the following assuming that $[F,H]$ is detectable.

(i) The strong solution exists and is unique.
(ii) If $[F,GQ^{1/2}]$ has no unreachable modes on the unit circle then the strong solution is also a stabilizing solution.
(iii) If $[F,GQ^{1/2}]$ has unreachable modes on the unit circle then there is no stabilizing solution.
(iv) If $[F,GQ^{1/2}]$ has unreachable modes on or inside the unit circle then there is the strong solution is positive semidefinite.
(iv) If $[F,GQ^{1/2}]$ has unreachable modes outside the unit circle then there are at least two positive semidefinite solutions to the ARE.

III. Scalar Systems

In this section we conduct a complete analysis of the scalar case in the absence of stabilizability and illustrate the results of [12]. In this case all system and covariance matrices are scalar, and $GQ^{1/2}=0$ and $H\neq 0$. Then the ARE becomes:

$$(1-F^2)P=-\frac{F^2P^2H^2}{R+PH^2}. \quad (11)$$

If $|F|\leq 1$ then the *only positive semidefinite solution to (11) is* $P=0$ *and the resulting Kalman gain is zero, and* $F-KH=F$ *is a pole of the filter.*

What are the implications of this? First if $|F|<1$, then $F-KH=F$ is automatically Schur and filter stability follows. The fact that $P=0$ also accords with intuition. In the absence of process noise $w[k]$, one expects the error covariance to converge to zero because of the law of large numbers.

The more intriguing case is when $F=1$. In this case $F-KH=F$ implies the steady state Kalman Filter is *not stable*. However, Theorem 1 indicates that the Kalman filter is still asymptoically stable though not eas. But how to interpret the fact that $P=0$? Recall that the Kalman filter is linear minimum variance optimal. If

$F=1$ then there is an L such that $|F-LH|<1$. Thus the linear filter,

$$\hat{x}[k+1]=(F-LH)\hat{x}[k]+L(y[k]-H\hat{x}[k]) \quad (12)$$

is expontially stable. Consequently zero process noise will lead to a steady state error covariance that is zero. The Kalman filter must match this steady state performance. Thus, even though the steady state Kalman filter is not stable, the solution of the ARE *still correctly predicts* the asymptotic performance.

Now suppose $|F|>1$. While $P=0$ remains a solution, there is potentially one more. In particular when $P\neq 0$, (11) reduces to:

$$\begin{aligned} & (1-F^2)(R+PH^2)=-F^2H^2P \\ \Leftrightarrow\quad & R+PH^2-F^2R=0 \\ \Leftrightarrow\quad & P=\frac{(F^2-1)R}{H^2}>0. \end{aligned} \quad (13)$$

Using (4) with $P\neq 0$, we find that the corresponding Kalman Gain K satisfies:

$$\begin{aligned} K &= \frac{FPH}{R+PH^2} \\ &= \frac{F^2-1}{FH} \end{aligned} \quad (14)$$

$$\begin{aligned} F-KH &= F-\frac{F^2-1}{F} \\ &= \frac{1}{F} \end{aligned} \quad (15)$$

which is stable iff $|F|>1$.

This directly shows that stabilizability is not necessary for the steady state filter to be stable. It is instructive that $P=0$ also solves (11). This is unsurprising as if the Kalman Filter is initialized with $P_0=0$ then the lack of process noise ensures that the steady state error covariance must be zero.

IV. A CONSTRUCTION

Recall that Matlab cannot solve the ARE without stabilizability. In this section we provide an algorithm that provides a positive semidefinite solution. First we provide a few facts. Consider two equivalent SVRs specifically,

$$A=TFT^{-1}, B=TGQ^{1/2}, C=HT^{-1}. \quad (16)$$

It is readily verified, and indeed well known, that if P solves the ARE (3) then Π that solves the following ARE

$$\begin{aligned} \Pi &= A\left(\Pi-\Pi C^\top\left(C\Pi C^\top+R\right)^{-1}C\Pi\right)A^\top \\ &+ BB^\top, \end{aligned} \quad (17)$$

obeys

$$\Pi=TPT^\top. \quad (18)$$

Further, $F-KH$ is Schur iff $A-\bar{K}C$ is Schur, where

$$\bar{K}=A\Pi C^\top\left(C\Pi C^\top+R\right)^{-1} \quad (19)$$

is the steady state Kalman gain of the transformed SVR.

Next suppose, $[F,GQ^{1/2}]$ is not completely stablizable. Then, [3], there exists a T such that in (16)

$$A=\begin{bmatrix}A_1 & A_2\\ 0 & A_3\end{bmatrix}, B=\begin{bmatrix}B_1\\ 0\end{bmatrix}, C=\begin{bmatrix}C_1 & C_2\end{bmatrix}, \quad (20)$$

where the matrices appearing in the partitioning, here and in the sequel, are compatibly dimensioned, the pair $[A_1,C_1]$ is completely detectable and $[A_1,B]$ is completely reachable. Lack of stabilizability means that A_3 has eigenvalues in the complement of the open unit disc.

As $[A_1,C_1]$ is completely detectable and $[A_1,B_1]$ completely reachable there is a unique $\Pi_1=\Pi_1^\top>0$ such that,

$$\begin{aligned} \Pi_1 &= A_1\left(\Pi_1-\Pi_1C_1^\top\left(C_1\Pi_1C_1^\top+R\right)^{-1}C\Pi_1\right) \\ & \quad A_1^\top+B_1B_1^\top. \end{aligned} \quad (21)$$

Then because of (20),

$$\Pi=\textbf{diag}\ \{\Pi_1,0\}, \quad (22)$$

solves (17) as

$$A\textbf{diag}\ \left\{\Pi_1-\Pi_1C_1^\top\left(C_1\Pi_1C_1^\top+R\right)^{-1}C_1\Pi_1,0\right\}A^\top \quad (23)$$

equals

$$\begin{bmatrix}A_1\left(\Pi_1-\Pi_1C_1^\top\left(C_1\Pi_1C_1^\top+R\right)^{-1}C_1\Pi_1\right)A_1^\top & 0\\ 0 & 0\end{bmatrix}$$

Thus Π in (22) solves (17). As Π_1 is positive semidefinite so is Π. And P obtained from (18) solves (3).

V. A SPECIAL CASE

We now consider a special case that is a direct generalization of the scalar example in the sense that it involves zero process noise i.e. $Q=0$ and all eigenvalues of F have magnitude greater than 1.

Theorem 2: Consider (1) and (2), with F^{-1} Schur, $Q=0$ and Assumption 1 in force. Then with $\Psi=\Psi^\top>0$ the unique symmetric solution of

$$F^\top\Psi F-\Psi=H^\top R^{-1}H, \quad (24)$$

$$P=\Psi^{-1} \quad (25)$$

solves (3). For this P, K in (4) obeys $F - KH \sim F^{-\top}$.

Proof: As all eigenvalues of F have magnitude greater than one, $[F, H]$ is completely observable. As $R > 0$ so is $[F, HR^{-1/2}]$. Thus, [1], (24) has a unique positive definite symmetric solution. Now under (25) and the matrix inversion lemma, [16], there holds:

$$\begin{aligned}
& F^\top \Psi F - \Psi = H^\top R^{-1} H \\
\Leftrightarrow\quad & F^\top P^{-1} F - P^{-1} = H^\top R^{-1} H \\
\Leftrightarrow\quad & F^\top P^{-1} F = P^{-1} + H^\top R^{-1} H \\
\Leftrightarrow\quad & \left(F^\top P^{-1} F\right)^{-1} = \left(P^{-1} + H^\top R^{-1} H\right)^{-1} \\
\Leftrightarrow\quad & F^{-1} P F^{-\top} = P - PH^\top \left(HPH^\top + R\right)^{-1} HP \\
\Leftrightarrow\quad & P = F\left(P - PH^\top \left(HPH^\top + R\right)^{-1} HP\right) F^\top.
\end{aligned}$$

This is indeed (3) with $Q = 0$.

Now observe under (25) and (4)

$$\begin{aligned}
F - KH &= F - FPH^\top \left(HPH^\top + R\right)^{-1} H \\
&= F\left(I - PH^\top \left(HPH^\top + R\right)^{-1} H)\right) \\
&= F\left(I + PH^\top R^{-1} H\right)^{-1} \\
&= F\left(I + P\left(F^\top P^{-1} F - P^{-1}\right)\right)^{-1} \\
&= PF^{-\top} P^{-1}.
\end{aligned}$$

This is clearly a direct and aesthetically appealing generalization of the scalar case when $|F| > 1$. In particular $F - KH$ has the same eigenvalues as $F^{-\top}$ and hence F^{-1}. Consequently $F - KH$ is Schur. Observe as in the scalar case when $Q = 0$, $P = 0$ also solves (3). Indeed as the process noise is zero, should one initialize (8) with $P[0] = 0$ then one is assuming that one knows the state trajectory, and $P = 0$ is the logical solution.

VI. Some uniqueness issues

Recall that [12] shows that should $[F, GQ^{1/2}]$ have an unreachable modes outside unit circle then the ARE has at least two positive semidefinite solutions. It also states that the strong solution is unique. What it does not answer is what happens if the only unreachable modes are in the closed unit disc? Specifically, can the ARE have multiple positive semidefinite solutions? Recall this is particularly important in the setting of [11] where all the unreachable modes are on the unit circle. Would in such a case the solution offered in Section IV be the only solution? In this section we answer this question in the affirmative.

With $\bar{G} = GQ^{1/2}$, define the controllability matrix:

$$\mathcal{C}(F, \bar{G}) = [\bar{G}, F\bar{G}, \cdots, F^{n-1}\bar{G}], \tag{26}$$

and call $\mathcal{N}(\mathcal{C}^\top(F, \bar{G}))$, the null space of $\mathcal{C}^\top(F, \bar{G})$. Make the following assumption that holds true of all discrete time systems obtained by sampling a continuous time system.

Assumption 2: The matrix $F \in \mathbb{R}^{n\times n}$ is nonsingular. In the sequel we will first characterize the null space of positive semidefinite solutions of (3). Recall that if an eigenvalue of F is unreachable then eigenvectors and/or generalized eigenvectors associated with it must lie in $\mathcal{N}(\mathcal{C}^\top(F, \bar{G}))$. We will first show that all eigenvectors and/or generalized eigenvectors associated with unreachable eigenvalues of $F^\top$ in the open unit disc are in the null space of P.

Lemma 1: Consider a positive semidefinite $P = P^\top$ that satisfies (3). Suppose the sequence,

$$\lim_{k\to\infty} z^\top F^k = 0.$$

and for all $k \in \mathbb{Z}_+$, $F^{\top k} z \in \mathcal{N}(\mathcal{C}^\top(F, \bar{G}))$. Then $Pz = 0$.

Proof:

From (3) and the Lemma hypothesis there holds:

$$\begin{aligned}
z^\top F^k P F^{\top k} z &\leq z^\top F^{k+1} P F^{\top(k+1)} z \\
&+ z^\top F^k \bar{G}\bar{G}^\top F^{\top k} z \\
&= z^\top F^{k+1} P F^{\top(k+1)} z.
\end{aligned}$$

In particular, this implies that for all $k \in \mathbb{Z}_+$,

$$z^\top P z \leq z^\top F^k P F^{\top k} z.$$

The result follows from the fact that the right hand side goes to zero as k tends to ∞.

The next lemma shows that under Assumption 1 eigenvectors associated with unreachable modes of $F^\top$ on the unit circle are also in the null space of P.

Lemma 2: Suppose, $R > 0$, $z \in \mathcal{N}(\mathcal{C}^\top(F, \bar{G}))$, for some $|\lambda| = 1$, $F^\top z = \lambda z$ and $[F, H]$ is completely detectable. Then $Pz = 0$.

Proof: From (3), and the hypothesis of the lemma:

$$z^\top PH\left(HPH^\top + R\right)^{-1} HPz = 0 \Leftrightarrow HPz = 0.$$

Further $Pz = FPF^\top z = \lambda FPz$. Then,

$$\begin{bmatrix} F - \lambda^* I \\ H \end{bmatrix} \lambda Pz = \begin{bmatrix} \lambda FPz - Pz \\ 0 \end{bmatrix} = 0.$$

Thus, unless $Pz = 0$, $[F, H]$ is not detectable.

The next lemma whose proof is omitted due to space restrictions shows that under Assumption 1 generalized eigenvectors associated with unreachable modes on the unit circle are also in the null space of P.

Recall that a matrix Φ has a chain of generalized eigenvectors z_i, $i \in \{0, 1, \ldots, g\}$, corresponding to the eigenvalue λ, if $\Phi z_0 = \lambda z_0$, and for all $i \in \{1, \cdots, g\}$, $\Phi z_i = \lambda z_i + z_{i-1}$.

Lemma 3: Suppose, $R > 0$, $z_i \in \mathcal{N}(\mathcal{C}^\top(F, \bar{G}))$, $i \in \{0, 1, \ldots, g\}$ are a chain of generalized eigenvectors of $F^\top$ corresponding to the eigenvalue λ. Suppose $|\lambda| = 1$, $[F, H]$ is completely detectable, and $P = P^\top \geq 0$ solves (3). Then for all $i \in \{0, 1, \ldots, g\}$

$$Pz_i = 0. \tag{27}$$

Lemmas 1- 3 help prove (proof omitted) the following.

Lemma 4: Suppose all unreachable poles of the SVR $\{F, \bar{G}, H\}$ are in the closed unit disc, and $P = P^\top \geq 0$ solves (3). Then there holds:

$$\mathcal{N}(\mathcal{C}^\top(F, \bar{G})) \subset \mathcal{N}(P). \tag{28}$$

This result shows that the dimension of the null space of P is no smaller than the number of unreachable modes in the closed unit disc. If these are the only unreachable modes, then as the nonnegative definite Π_1 solving (21) is unique and positive definite, the only positive semidefinite solution of (17) under (20) and (16) is the Π in (22). Thus in view of (18) we have the following uniqueness result.

Theorem 3: Suppose assumptions 1 and 2 hold, $[F, \bar{G}]$ is not completely reachable but all modes outside the unit circle are reachable. Then the positive semidefinite solution of (3) is unique.

VII. Conclusion

We have considered the solution of the filtering ARE in the absence of a stabilizability condition assumed in most references and have provided a constructive positive semidefinite solution in the absence of stabilizability. We have shown that this solution is unique if the unreachable modes are in the closed unit disc. Though our results are derived in discrete time, we believe they extend to continuous time with the unit circle obviously replaced by the imaginary axis.

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