Random Symmetric Gossip Consensus with Overhearing Nodes in Wireless Networks

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***Abstract*—This paper considers consensus in wireless networks with random symmetric gossip. During each timeslot, a pair of nodes is randomly selected. These nodes then exchange messages and adjust their states to reduce the distance from consensus in the network. Since only two messages are exchanged and only two nodes adjust their states in each timeslot, the convergence rate of symmetric gossip to a consensus state is often prohibitively slow (especially for networks with a large number of nodes). This paper proposes an extension to the symmetric gossip consensus model which takes advantage of the broadcast nature of wireless transmissions. Specifically, this paper considers a scenario in which some nodes can "overhear" the transmissions of the randomly selected node pair in each timeslot. These nodes can use this overheard information to adjust their states for "free", i.e., without any additional messaging in each timeslot, consequently improving convergence rates. This paper analyzes the performance of random symmetric gossip with overhearing nodes under several different network topologies. Numerical results are also provided for consensus synchronization scenario. These results show that overhearing nodes can significantly improve the convergence rate of symmetric gossip systems without increasing messaging overhead.**

***Index Terms*—consensus, symmetric gossip, wireless networks, synchronization**

I. Introduction

Random consensus methods have been considered in [1]–[15] and are attractive since consensus functions, e.g., distributed estimation or synchronization, can be embedded in existing network traffic with random information flows. This paper considers the *random symmetric gossip* consensus problem. In each timeslot, a pair of nodes is randomly selected according to some distribution and exchange messages while all other nodes remain silent. Each node in the randomly selected pair uses the information from the other node in the pair to adjust their state and improve the consensus metric of the network. In this paper, we adopt a "distance from consensus" metric, which is a measure of the state displacement from the current average. Given a state $\boldsymbol{x}[k] = [x_1[k], \dots, x_N[k]]^\top \in \mathbb{R}^N$ at time k, the distance from consensus at time k is defined as

$$d[k] := \frac{1}{N}\|\boldsymbol{x}[k] - \mathbf{1}_N \bar{x}[k]\|_2^2 \tag{1}$$

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where $\bar{x}[k] := \frac{1}{N}\sum_{i=1}^N x_i[k]$ and $\mathbf{1}_N \in \mathbb{R}^N$ is a vector of ones. In the context of synchronization, the state $\boldsymbol{x}[k]$ corresponds to the drifts and/or offsets of the clocks in the network. It has been shown that, under certain conditions, the distance from consensus converges almost surely in a mean squared sense at an exponential rate, e.g., $\mathbb{P}\left\{\lim_{k\to\infty}(d[k])^{1/k} = \lambda^2\right\} = 1$ for a given constant λ [9], [10]. In asymmetric and symmetric gossip systems, however, the convergence rate can be prohibitively slow (especially for systems with a large number of nodes) due to the fact that only one or two nodes adjust their states in each timeslot. Random broadcast consensus [15] can improve the convergence rate of consensus systems at the cost of increased messaging overhead in each timeslot.

In this paper, we propose an extension of the random symmetric gossip consensus model which takes advantage of the broadcast nature of wireless transmissions, potentially providing convergence rates similar to random broadcast without the additional messaging overhead of random broadcast. Actually, the "overhearing" strategy has been studied in some prior literatures and has wide applications. In [16], the authors studied a two-hop interference channel with the help of two relay nodes and assumed that the transmissions of the relays in the second hop can be overheard by the sources in the first hop. Strategies were designed to optimally consolidate the dual role of the relay signal: conveying the message to its corresponding destination versus sending a feedback signal to increase the capacity of the first hop. In [17], the authors studied the naming game (NG), which describes the agreement dynamics of a population of N agents interacting locally in pairs leading to the emergence of a shared vocabulary, and introduce the concept of overhearing. Results showed that the population of agents reaches a faster agreement with a significantly low-memory requirement with overhearing strategy.

Specifically, this paper considers a scenario called "random symmetric gossip with overhearing nodes" in which some nodes can "overhear" the transmissions of the randomly selected node pair in each timeslot. In timeslot k, when node i transmits a message to node j and then node j transmits a response to node i, other nodes who are in the vicinity of node i and node j may overhear one or both of these messages. Using this "free" information, these overhearing nodes can also adjust their states to improve the consensus metric. Intuitively, the convergence rate is improved since more nodes adjust their states in each timeslot.

This paper studies the convergence rate of random symmetric gossip with overhearing nodes in three scenarios: (1) fully-connected networks, (2) line or ring networks, and (3) uniformly distributed planar networks with fixed overhearing radius. Numerical results are also provided to demonstrate the performance of random symmetric gossip with overhearing nodes and convergence rates are compared to conventional random symmetric gossip systems without overhearing nodes as well as random broadcast consensus systems [15]. The results show that random symmetric gossip with overhearing nodes can provide a convergence rate similar to random broadcast consensus without the additional messaging overhead associated with random broadcast consensus. In fact, comparing the convergence rates from the point of view of number of message exchanges, our results show that random symmetric gossip with overhearing nodes outperforms random broadcast consensus in the scenarios considered.

II. System Model

We assume a connected time-slotted wireless network with N nodes denoted as $\mathcal{N} = \{1, \ldots, N\}$. The topology of the network is assumed to be fixed. In each timeslot, the node pair (i, j) is randomly selected and exchange messages. We denote

$$\mathcal{E} = \left\{(i,j) \in \mathcal{N}^2 \,:\, i < j \text{ and } (i,j) \text{ can exchange messages}\right\}$$

as the set of all distinct node pairs in the network. Due to our focus on *symmetric* gossip, we do not distinguish between the pair (i, j) and the pair (j, i) and henceforth assume $i < j$. We assume that each node pair is selected in each timeslot independently and with equal probability. Specifically,

$$\text{Prob}\{(i,j) \text{ selected}\} = \begin{cases} p & (i,j) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

where $p = \frac{1}{|\mathcal{E}|}$. The sets $\mathcal{O}_i$ and $\mathcal{O}_j$ denote the sets of nodes able to overhear node i and node j, respectively. Note that $\mathcal{O}_i$ and $\mathcal{O}_j$ are not disjoint in general. We assume that each node in the sets $\mathcal{O}_i$ and $\mathcal{O}_j$ has a probability q of overhearing the transmissions from node i and node j, respectively. Setting $q = 0$ prevents overhearing and reduces the system model to the conventional symmetric gossip case without overhearing nodes.

The following sections analyze the behavior of random symmetric gossip with overhearing nodes in three scenarios: (1) fully-connected networks, (2) line or a ring networks, and (3) uniformly distributed planar networks with fixed overhearing radius.

A. Fully-Connected Network

In this type of network, all of the nodes in the network are located close enough to each other such that any pair of nodes has the ability to exchange messages. The set of distinct node pairs is thus

$$\mathcal{E} = \{(1,2), \ldots, (1,N), (2,3), \ldots, (2,N), \ldots, (N-1,N)\}$$

with $|\mathcal{E}| = \frac{N^2-N}{2}$. The set of potential overhearing nodes of node i is $\mathcal{O}_i = \mathcal{N} \setminus \{i, j\}$ and the set of potential overhearing nodes of node j is $\mathcal{O}_j = \mathcal{N} \setminus \{i, j\} = \mathcal{O}_i$.

B. Line or Ring Network

In this scenario, all N nodes are located on a line or a ring. In each timeslot, it is assumed that only two adjacent nodes can exchange messages.

For a line network, the set of distinct node pairs is given as

$$\mathcal{E} = \{(1,2), (2,3), \ldots, (N-1,N)\}$$

with $|\mathcal{E}| = N - 1$. The set of potential overhearing nodes of node i and node $j = i + 1$ can be expressed respectively as

$$\mathcal{O}_i = \begin{cases} \varnothing & i = 1 \\ \{i-1\} & \text{otherwise} \end{cases}$$

and

$$\mathcal{O}_{i+1} = \begin{cases} \varnothing & i = N-1 \\ \{i+2\} & \text{otherwise.} \end{cases}$$

A ring network is similar to the line network except that node 1 can also exchange messages with node N. In this case, the set of distinct node pairs is given as

$$\mathcal{E} = \{(1,2), (2,3), \ldots, (N-1,N), (1,N)\}$$

with $|\mathcal{E}| = N$. The set of potential overhearing nodes of node i and node $j = i + 1$ can be expressed respectively as

$$\mathcal{O}_i = \begin{cases} \{N\} & i = 1 \\ \{i-1\} & \text{otherwise} \end{cases}$$

and

$$\mathcal{O}_{i+1} = \begin{cases} \{1\} & i = N-1 \\ \{i+2\} & \text{otherwise.} \end{cases}$$

C. Uniform Planar Network With Fixed Overhearing Radius

In this scenario, all nodes are uniformly distributed in a fixed region, say a rectangle with length a and width b. If we use (x_i, y_i) to denote the coordinates of the ith node, then we have $x_i \sim U(0, a)$ and $y_i \sim U(0, b)$ for all $i \in \mathcal{N}$. We also assume that all coordinates are independently generated. We assume that node i can only exchange messages with nodes inside a circle with center (x_i, y_i) and radius c. If we use $\rho(i, j)$ to denote the distance between node i and node j, then the set of nodes able to exchange messages with node i can be expressed as $\mathcal{E}_i = \{j \in \mathcal{N} \setminus \{i\} : \rho(i, j) \leq c\}$. The set of distinct node pairs follows as

$$\mathcal{E} = \bigcup_{i=1}^{N} \{(i,j) \in \mathcal{N}^2 : j > i \text{ and } \rho(i,j) \leq c\}.$$

We also assume that only the nodes inside a circle with center (x_i, y_i) and radius $r \leq c$ can overhear node i. Thus the set of nodes which can potentially overhear node i can be expressed as $\mathcal{O}_i = \{\ell \in \mathcal{E}_i \setminus \{j\} : \rho(i, \ell) \leq r\}$ and the set of nodes which can potentialy overhear node j can be expressed as $\mathcal{O}_j = \{\ell \in \mathcal{E}_j \setminus \{i\} : \rho(j, \ell) \leq r\}$.

III. Consensus Dynamics

In this section, we describe the consensus dynamics of random symmetric gossip with overhearing nodes. In general, the state vector $\boldsymbol{x}[k] \in \mathbb{R}^N$ has random dynamics governed by

$$\boldsymbol{x}[k+1] = (\boldsymbol{I}_N + \mu \boldsymbol{R}[k])\boldsymbol{x}[k] \tag{2}$$

where $\boldsymbol{I}_N$ is an $N \times N$ identity matrix, $\mu > 0$ is a stepsize parameter, and $\boldsymbol{R}[k] \in \mathbb{R}^{N\times N}$ is a matrix randomly drawn from some finite set $\mathcal{R}$ satisfying the property $\boldsymbol{R}[k]\mathbf{1}_N = \mathbf{0}$ for all $\boldsymbol{R}[k] \in \mathbb{R}$ where $\mathbf{1}_N$ is an $N \times 1$ vector of ones. In the context of synchronization, the state vector $\boldsymbol{x}[k]$ corresponds to clock drifts and/or offsets and the $\boldsymbol{R}[k]$ matrices correspond to clock corrections resulting from random interactions among the nodes in the network.

In a conventional symmetric gossip system, conditioning on the node pair (i, j), the update matrix $\boldsymbol{R}[k]$ can be expressed as

$$\boldsymbol{R}[k] = \boldsymbol{e}_i(\boldsymbol{e}_j - \boldsymbol{e}_i)^\top + \boldsymbol{e}_j(\boldsymbol{e}_i - \boldsymbol{e}_j)^\top$$

where $\boldsymbol{e}_i$ is the i^{th} standard basis vector with all elements equal to zero except the i^{th} element which is equal to one. In the case of random symmetric gossip with overhearing nodes, the update matrix $\boldsymbol{R}[k]$ can be expressed as

$$\begin{aligned}\boldsymbol{R}[k] &= \boldsymbol{e}_i(\boldsymbol{e}_j - \boldsymbol{e}_i)^\top + \boldsymbol{e}_j(\boldsymbol{e}_i - \boldsymbol{e}_j)^\top \\ &\quad + \sum_{m\in\mathcal{Q}_i} \boldsymbol{e}_m(\boldsymbol{e}_i - \boldsymbol{e}_m)^\top + \sum_{n\in\mathcal{Q}_j} \boldsymbol{e}_n(\boldsymbol{e}_j - \boldsymbol{e}_n)^\top\end{aligned}$$

where $\mathcal{Q}_i \subseteq \mathcal{O}_i$ and $\mathcal{Q}_j \subseteq \mathcal{O}_j$ are the nodes that overhear the messages from node i and node j, respectively. Although $\mathcal{O}_i$ and $\mathcal{O}_j$ are deterministic and fixed, the sets $\mathcal{Q}_i$ and $\mathcal{Q}_j$ are random and depend on the overhearing probability parameter q. Also note that, even though we are considering a symmetric gossip system from the perspective of nodes i and j, from the perspective of the overhearing nodes the information flow is asymmetric.

For random broadcast [15], in each timeslot each node chooses equiprobably whether to transmit or receive. Hence, the average number of messages transmitted in each timeslot is $\frac{N}{2}$. Assuming a fully-connected network so that all receiving nodes receive the transmissions of all transmitting nodes and letting $\mathcal{T}[k] \subseteq \mathcal{N}$ denote the set of transmitting nodes in timeslot k, the update matrix $\boldsymbol{R}[k]$ can be expressed as

$$\boldsymbol{R}[k] = \sum_{j\in\mathcal{N}\setminus\mathcal{T}[k]} \sum_{i\in\mathcal{T}[k]} \boldsymbol{e}_j(\boldsymbol{e}_i - \boldsymbol{e}_j)^\top.$$

IV. Numerical Results

This section presents numerical results demonstrating the effectiveness of random symmetric gossip with overhearing nodes in the context of drift synchronization in a wireless communication network. Specifically, the goal is to synchronize the frequencies of all of the clocks in the network through consensus techniques. In each timeslot, the randomly selected node pair (i, j) exchange messages, estimate their relative clock drifts, and then adjust their drifts to improve the consensus metric as discussed in [18]. The distance from consensus metrics is computed for the drifts in each timeslot and ensemble averaged over 1000 runs. In all of the results in this section, the initial drifts $x_i[0]$ for $i = 1, \dots, N$ were randomly generated as i.i.d. zero mean Gaussian random variables with standard deviation 100 μs/timeslot.

A. Fully-Connected Network

Figure 1 shows the ensemble averaged drift distance from consensus metric for $N = 10$ and $N = 100$ node networks with stepsize $\mu = 0.5$ and varying overhearing probabilities $q \in \{0, 0.1, 0.25, 0.5\}$ in fully-connected network. The $q = 0$ case corresponds to conventional random symmetric gossip. It is observed that when the overhearing probability q increases, the convergence rate of the distance from consensus metric also increases. We also observe that when the overhearing probability q increases, the drift convergence rate gap between the system with a small number of nodes ($N = 10$) and the system with a large number of nodes ($N = 100$) decreases due to the large number of overhearing nodes in the $N = 100$ case. In Figure 1, we also show the ensemble averaged drift distance from consensus metric for random broadcast [15] with stepsizes $\mu \in \{0.1, 0.25\}$ for an $N = 10$ node system. Observe that the symmetric gossip system with overhearing probability $q = 0.5$ has a better convergence rate than the random broadcast system.

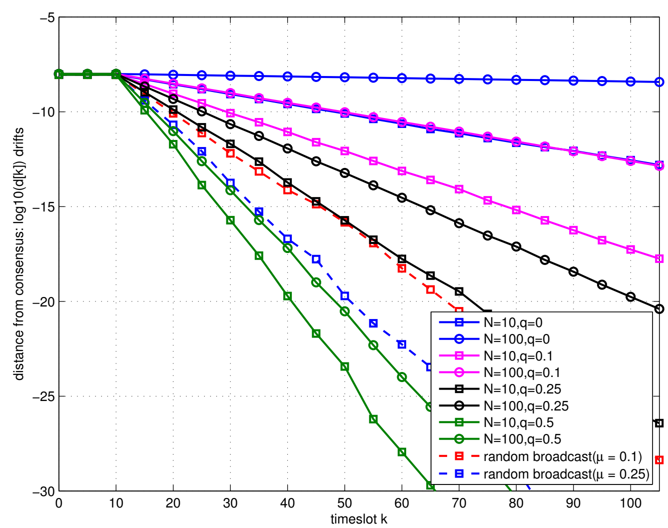

Fig. 1. Average drift distance from consensus metrics for $N = 10, 100$ node network with stepsize $\mu = 0.5$ and with different overhearing probability $q \in \{0, 0.1, 0.25, 0.5\}$ in fully-connected network. Drift compensation begins at timeslot $k = 10$.

Figure 2 shows the ensemble averaged drift distance from consensus metric versus *the number of message exchanges* for $N = 10$ node network with stepsize $\mu = 0.5$ and with varying overhearing probabilities $q \in \{0, 0.1, 0.25, 0.5\}$. Recall, for random symmetric gossip with overhearing nodes, the number of messages transmitted in each timeslot is always two. For the random broadcast system, in each timeslot each node is selected as a transmitter or a receiver equiprobably. Hence the number of messages in each timeslot is binomially distributed with probability $\frac{1}{2}$ and has a mean of $\frac{N}{2}$. In this example with $N = 10$, the average number of messages transmitted in each timeslot is five. These results show that random symmetric gossip with overhearing nodes significantly outperforms random broadcast consensus from the point of view of number of message exchanges.

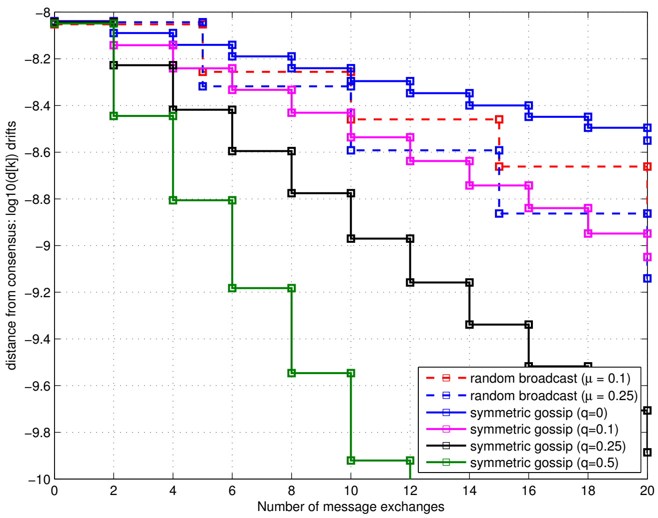

Fig. 2. Average drift distance from consensus metrics for N=10 node network scenarios with stepsize $\mu = 0.5$ and with different overhearing probability $q \in \{0, 0.1, 0.25, 0.5\}$ versus the number of message exchanges in fully-connected network.

B. Line or Ring Network

Figure 3 shows the ensemble averaged drift distance from consensus metric for $N = 10$ and $N = 100$ node networks with stepsize $\mu = 0.5$ and varying overhearing probabilities $q \in \{0, 0.1, 0.25, 0.5\}$. Unlike the fully-connected network, the convergence rate of random symmetric gossip with overhearing nodes is not significantly improved with respect to conventional random symmetric gossip since there are at most two nodes that can overhear in each timeslot.

C. Uniform Planar Network With Fixed Overhearing Radius

Figure 4 shows the ensemble averaged drift distance from consensus metric for $N = 10$ and $N = 100$ node networks with stepsize $\mu = 0.5$ and varying overhearing probabilities $q \in \{0, 0.1, 0.25, 0.5\}$ in a uniformly distributed planar network with fixed overhearing radius. These results are for one realization of the node locations (which were confirmed to form a weakly connected graph) and are typical of the results achieved with other realizations of the node locations. We assume that the planar region is a square with length $a = 100$ m and width $b = 100$ m. We further assume that the messaging radius is $c = 100$ m and the overhearing radius is $r = 50$ m. From Figure 4, it is observed that the drift convergence rate improvement of the system with large number of nodes ($N = 100$) is larger than that of the system with small number of nodes ($N = 10$). Intuitively, this is due to the fact that when each node has a limited

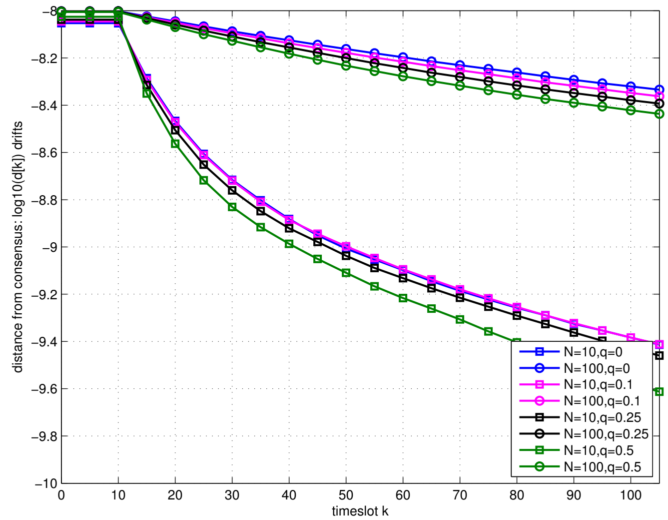

Fig. 3. Average drift distance from consensus metrics for $N = 10, 100$ node network with stepsize $\mu = 0.5$ and with different overhearing probability $q \in \{0, 0.1, 0.25, 0.5\}$ in line network. Drift compensation begins at timeslot $k = 10$.

overhearing range, increasing the number of nodes in the system consequently increases the number of nodes able to overhear the messages exchanged in each timeslot.

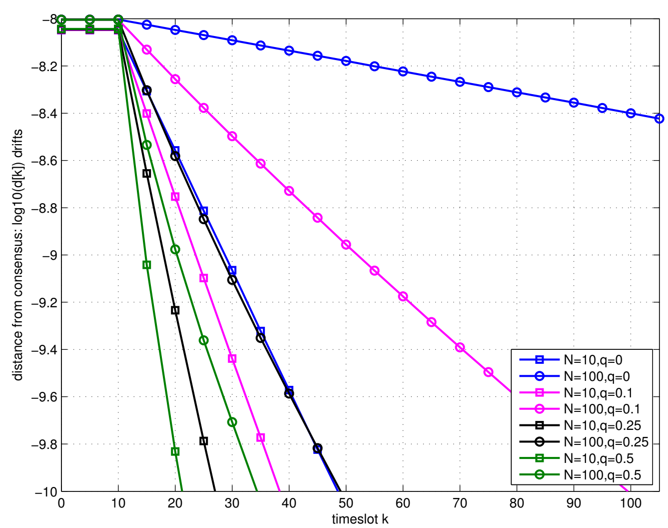

Fig. 4. Average drift distance from consensus metrics for $N = 10, 100$ node network with stepsize $\mu = 0.5$ and with different overhearing probability $q \in \{0, 0.1, 0.25, 0.5\}$ in uniformly distributed planar network with fixed overhearing radius. Drift compensation begins at timeslot $k = 10$.

V. Conclusion

This paper studies random symmetric gossip with overhearing nodes in wireless networks. Numerical results show that random symmetric gossip with overhearing nodes can improve the convergence rate of symmetric gossip systems with significant gains occurring when the network is highly-connected. Particularly, in fully-connected networks, overhearing nodes can significantly improve the convergence rate of symmetric gossip systems and provide convergence rates better than random broadcast. A particularly appealing feature of random symmetric gossip with overhearing nodes is that these gains are achieved without any increase in messaging overhead.

While this paper demonstrated the effectiveness of random symmetric gossip with overhearing nodes via numerical results, it is also of interest to develop analytical results for the optimum stepsize and bounds on the achievable convergence rates. In the case of planar networks with randomly deployed nodes, it may also be of interest to study the relationship between messaging *energy* and convergence rate since decreasing messaging energy leads to smaller messaging and overhearing radii and, hence, slower convergence to consensus.

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