Digital Signal Processing
Minimum-Phase All-Pass Decomposition

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Minimum-Phase All-Pass Decomposition

Suppose we have a causal stable rational transfer function $H(z)$ with one or more zeros outside the unit circle. We denote the zeros outside the unit circle as $\{c_1, \ldots, c_M\}$.

We can form a minimum phase system with the same magnitude response by reflecting these poles to their conjugate symmetric locations inside the unit circle, i.e.,

$$H_{\text{min}}(z) = H(z) \cdot \frac{z^{-1} - c_1^*}{1 - c_1 z^{-1}} \cdots \frac{z^{-1} - c_M^*}{1 - c_M z^{-1}}$$

unit-magnitude all-pass filter

It should be clear that $H_{\text{min}}(z)$ and $H(z)$ have the same magnitude response. Moreover, we have the decomposition

$$H(z) = H_{\text{min}}(z) \cdot \left(\frac{z^{-1} - c_1^*}{1 - c_1 z^{-1}} \cdots \frac{z^{-1} - c_M^*}{1 - c_M z^{-1}}\right)^{-1} = H_{\text{min}}(z)H_{\text{ap}}(z)$$

where $H_{\text{ap}}(z)$ is an all-pass filter.
Example 1 (part 1 of 2)

Suppose

\[ H(z) = \frac{1 - 2z^{-1}}{1 + \frac{1}{3}z^{-1}} \]

This system is clearly not minimum phase since it has a zero at \( z = 2 \). We can reflect this zero inside the unit circle with an all-pass filter to write

\[ H_{\text{min}}(z) = H(z) \cdot \frac{z^{-1} - 2}{1 - 2z^{-1}} \]

\[ = \frac{z^{-1} - 2}{1 + \frac{1}{3}z^{-1}} \]

hence

\[ H(z) = \frac{z^{-1} - 2}{1 + \frac{1}{3}z^{-1}} \cdot \frac{1 - 2z^{-1}}{z^{-1} - 2} \]

\[ = H_{\text{min}}(z) \cdot H_{\text{ap}}(z) \]
Example 1 (part 2 of 2)

Note that, just requiring the zeros to be inside the unit circle does not uniquely specify $H_{\text{min}}(z)$. For example, we could also write

$$H(z) = \frac{2 - z^{-1}}{1 + \frac{1}{3}z^{-1}} \cdot \frac{2z^{-1} - 1}{z^{-1} - 2}$$

where $G = -H$ for both systems. Does it matter which one we pick?

To satisfy the minimum phase delay property, recall that we require $\angle H_{\text{min}}(e^{j0}) = 0$. Note that

$$H_{\text{min}}(e^{j0}) = \frac{1 - 2}{1 + \frac{1}{3}} < 0 \text{ and } G_{\text{min}}(e^{j0}) = \frac{2 - 1}{1 + \frac{1}{3}} > 0$$

hence $\angle H_{\text{min}}(e^{j0}) = -\pi$ and $\angle G_{\text{min}}(e^{j0}) = 0$. The correct answer is to choose the $G_{\text{min}}(z)G_{\text{ap}}(z)$ decomposition.
Suppose

\[ H(z) = \frac{(1 + 3z^{-1})(1 - \frac{1}{2}z^{-1})}{z^{-1}(1 + \frac{1}{3}z^{-1})} \]

This is also clearly not minimum phase due to the zero at \( z = -3 \). We can reflect this zero inside the unit circle to write

\[ H_{\text{min}}(z) = H(z) \cdot \frac{z^{-1} + 3}{1 + 3z^{-1}} \]

\[ = \frac{(z^{-1} + 3)(1 - \frac{1}{2}z^{-1})}{z^{-1}(1 + \frac{1}{3}z^{-1})} \]

\[ = \frac{3(1 - \frac{1}{2}z^{-1})}{z^{-1}} \]

Recall that minimum phase filters must be causal. Since \( H_{\text{min}}(z) = 3z - \frac{1}{2} \) is clearly not causal, we can factor out the \( z^{-1} \) denominator term (putting it in the all-pass filter since it does not affect the magnitude response) to arrive at

\[ H(z) = 3 \left( 1 - \frac{1}{2}z^{-1} \right) \cdot \frac{z^{-1} + 3}{z^{-1}(1 + 3z^{-1})} \]

\[ H(z) = \underbrace{H_{\text{min}}(z)} \underbrace{H_{\text{ap}}(z)} \]
Equalization of Nonminimum Phase Channel

Suppose \( H_1(z) = \frac{(z-4)(z+5)}{(z+0.5)(z-0.3)} \) with ROC \(|z| > 0.5\).

We form the inverse system \( H_2(z) = \frac{(z+0.5)(z-0.3)}{(z-4)(z+5)} \). Note that there is no causal stable inverse here.

One approach in this case is to factor \( H_1(z) \) into a causal stable minimum phase filter and a causal stable allpass filter, i.e.

\[
H_1(z) = H_{\text{min}}(z)H_{\text{ap}}(z) = \frac{(4z-1)(5z+1)}{(z+0.5)(z-0.3)} \quad \frac{(z-4)(z+5)}{(4z-1)(5z+1)}
\]

and invert just the minimum phase component, i.e. \( H_2(z) = \frac{(z+0.5)(z-0.3)}{(4z-1)(5z+1)} \).

Then \( H_1(z)H_2(z) \neq 1 \) but rather \( H_1(z)H_2(z) = H_{\text{ap}}(z) \). Hence, the equalizer \( H_2(z) \) corrects the magnitude distortion, but leaves some residual phase distortion.