Digital Signal Processing
FIR Filters with Generalized Linear Phase

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Generalized Linear Phase

Definition
A system has **linear phase** if its phase response \( \theta(\omega) = \angle H(e^{j\omega}) = -c\omega \) for all \( \omega \) and any constant \( c \).

In general, a linear phase system has frequency response

\[
H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\omega c}
\]

and delays all frequencies by the same amount of time.

Definition
A system has **generalized linear phase** if its frequency response can be written as \( H(e^{j\omega}) = A(e^{j\omega})e^{-j\omega c + jb} \) where \( c \) and \( b \) are constants and \( A(e^{j\omega}) \) is a real (not necessarily positive) function.

GLP systems are sometimes called “affine phase” systems and have constant group delay except at discontinuities in the phase response.
Type I Causal FIR Generalized Linear-Phase Systems

Characteristics:
- Odd number of coefficients.
- Symmetric

Example (filter order $M = 2$): $h[n] = \{1, 2, 1\}$.

Frequency response

\[
H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n}
\]

\[
= h[0](e^{-j\omega 0} + e^{-j\omega M}) + h[1](e^{-j\omega 1} + e^{-j\omega (M-1)}) + \cdots + h[M/2]e^{-j\omega M/2}
\]

\[
= e^{-j\omega M/2} \left( h[0](e^{j\omega M/2} + e^{-j\omega M/2}) + \cdots + h[M/2] \right)
\]

\[
= e^{-j\omega M/2} \left( h[0] \cdot 2 \cos(\omega M/2) + h[1] \cdot 2 \cos(\omega (M/2 - 1)) \cdots + h[M/2] \right)
\]

\[
= e^{-j\omega M/2} \sum_{k=0}^{M/2} a_1[k] \cos(\omega k)
\]

where

\[
a_1[k] = \begin{cases} 
  h[M/2] & k = 0 \\
  2h[M/2 - k] & \text{otherwise}.
\end{cases}
\]
Type II Causal FIR Generalized Linear-Phase Systems

Characteristics:

- Even number of coefficients.
- Symmetric

Example (filter order $M = 3$): $h[n] = \{1, 2, 2, 1\}$.

Frequency response

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n}$$

$$= h[0](e^{-j\omega 0} + e^{-j\omega M}) + \cdots + h[(M - 1)/2](e^{-j\omega(M-1)/2} + e^{-j\omega(M+1)/2})$$

$$= e^{-j\omega M/2} \left( h[0](e^{j\omega M/2} + e^{-j\omega M/2}) + \cdots + h[(M - 1)/2](e^{j\omega/2} + e^{-j\omega/2}) \right)$$

$$= e^{-j\omega M/2} \left( h[0] \cdot 2 \cos(\omega M/2) + \cdots + h[(M - 1)/2] \cdot 2 \cos(\omega/2) \right)$$

$$= e^{-j\omega M/2} \sum_{k=0}^{(M-1)/2} a_2[k] \cos \left( \omega \left( k + \frac{1}{2} \right) \right)$$

where

$$a_2[k] = 2h[(M - 1)/2 - k].$$
Type III Causal FIR Generalized Linear-Phase Systems

Characteristics:

- Odd number of coefficients.
- Anti-symmetric

Example (filter order $M = 2$): $h[n] = \{1, 0, -1\}$ ($h[M/2]$ must be zero).

Frequency response

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n] e^{-j\omega n}$$

$$= h[0](e^{-j\omega 0} - e^{-j\omega M}) + \cdots + h[1](e^{-j\omega 1} - e^{-j\omega (M-1)}) + \cdots + h[M/2]e^{-j\omega M/2}$$

$$= e^{-j\omega M/2} \left( h[0](e^{j\omega M/2} - e^{-j\omega M/2}) + h[1](e^{j\omega (M/2-1)} - e^{-j\omega (M/2-1)}) + \cdots \right)$$

$$= e^{-j\omega M/2} (h[0] \cdot 2j \sin(\omega M/2) + h[1] \cdot 2j \sin(\omega (M/2 - 1)) + \cdots)$$

$$= je^{-j\omega M/2} \sum_{k=0}^{M/2-1} a_3[k] \sin(\omega(k + 1))$$

where

$$a_3[k] = 2h[M/2 - k - 1].$$
Type IV Causal FIR Generalized Linear-Phase Systems

Characteristics:
- Even number of coefficients.
- Anti-symmetric

Example (filter order $M = 3$): $h[n] = \{1, -2, 2, -1\}$.

Frequency response

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n}$$

$$= h[0](e^{-j\omega} - e^{-j\omega M}) + \cdots + h[(M - 1)/2](e^{-j\omega(M-1)/2} - e^{-j\omega(M+1)/2})$$

$$= e^{-j\omega M/2} \left( h[0](e^{j\omega M/2} - e^{-j\omega M/2}) + \cdots + h[(M - 1)/2](e^{j\omega/2} - e^{-j\omega/2}) \right)$$

$$= e^{-j\omega M/2} \left( h[0] \cdot 2j \sin(\omega M/2) + \cdots + h[(M - 1)/2] \cdot 2j \sin(\omega/2) \right)$$

$$= je^{-j\omega M/2} \sum_{k=0}^{(M-1)/2} a_4[k] \sin \left( \omega \left( k + \frac{1}{2} \right) \right)$$

where

$$a_4[k] = 2h[(M - 1)/2 - k]$$
Summary

<table>
<thead>
<tr>
<th>Type</th>
<th>Impulse response symmetry</th>
<th>Impulse response length</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>symmetric</td>
<td>$M + 1$ odd (order is even)</td>
</tr>
<tr>
<td>II</td>
<td>symmetric</td>
<td>$M + 1$ even (order is odd)</td>
</tr>
<tr>
<td>III</td>
<td>antisymmetric</td>
<td>$M + 1$ odd (order is even)</td>
</tr>
<tr>
<td>IV</td>
<td>antisymmetric</td>
<td>$M + 1$ even (order is odd)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Type</th>
<th>LPF</th>
<th>HPF</th>
<th>BPF</th>
<th>BSF</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Most versatile.</td>
</tr>
<tr>
<td>II</td>
<td>Y</td>
<td>N</td>
<td>Y</td>
<td>N</td>
<td>Zero at $z = -1$.</td>
</tr>
<tr>
<td>III</td>
<td>N</td>
<td>N</td>
<td>Y</td>
<td>N</td>
<td>Zeros at $z = \pm 1$.</td>
</tr>
<tr>
<td>IV</td>
<td>N</td>
<td>Y</td>
<td>Y</td>
<td>N</td>
<td>Zero at $z = 1$.</td>
</tr>
</tbody>
</table>

All have the same constant group delay, $\tau_g(\omega) = \frac{M}{2}$, which is an integer for Type I and Type III but is not an integer for Type II and Type IV.