ECE504 Homework Assignment Number 2
Due by 8:50pm on 30-Sep-2008

Tips: Make sure your reasoning and work are clear to receive full credit for each problem.

1. 3 pts. For \( Q \in \mathbb{C}^{n \times n} \) and \( \alpha \in \mathbb{C} \), compute \( \det(\alpha Q) \) and \( \text{adj}(\alpha Q) \) in terms of \( \alpha, \det(Q) \) and \( \text{adj}(Q) \). What does this say about \( (\alpha Q)^{-1} \) for \( \alpha \neq 0 \) and \( \det(Q) \neq 0 \)?

2. 4 pts. Suppose you are given a lumped discrete-time LTI system described by the state space equations
   \[
   x[k+1] = Ax[k] + Bu[k] \\
   y[k] = Cx[k] + Du[k].
   \]  
   \( (1) \)
   Now suppose we define \( v[k] = Px[k] \) for all \( k \) where \( P \in \mathbb{R}^{n \times n} \) is known and \( P \) is invertible such that \( x[k] = P^{-1}v[k] \).
   
   (a) Find \( \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \), in terms of \( A, B, C, D, \) and \( P \) such that \( v[k+1] = \tilde{A}v[k] + \tilde{B}u[k] \)
   \[
   y[k] = \tilde{C}v[k] + \tilde{D}u[k].
   \]  
   \( (2) \)
   (b) Show that \( (1) \) and \( (2) \) have the same transfer function by showing that
   \[
   \tilde{C}(zI - \tilde{A})^{-1}\tilde{B} + \tilde{D} = C(zI - A)^{-1}B + D.
   \]
   Hint: Recall that \( (XY)^{-1} = Y^{-1}X^{-1} \).

3. 6 pts. Suppose \( A = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 \\
-a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1}
\end{bmatrix}, \)
   \( B = [0,0,\ldots,0,1]^\top, \)
   \( C = [b_0,b_1,\ldots,b_{n-1}], \) and \( D = 0. \) Also suppose that
   \[
   \hat{g}(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0},
   \]
   (a) Is \( \{A,B,C,D\} \) is a state-space realization for the single-input single-output transfer function \( \hat{g}(s) \)?
   (b) Let \( \tilde{A} = A^\top, \tilde{B} = C^\top, \tilde{C} = B^\top, \) and \( \tilde{D} = D. \) Show that \( \{\tilde{A},\tilde{B},\tilde{C},\tilde{D}\} \) is a state-space realization for the single-input single-output transfer function \( \hat{g}(s) \).
   (c) Find two different state-space realizations for \( \hat{g}(s) = \frac{s^3}{s^3 + 2s^2 - s + 2} \).
   (d) Find two different state-space realizations for the discrete time system \( \hat{g}(z) = \frac{z^{-1}}{z^{-2} + 2z^{-1} - 3} \).
4. 6 pts. Consider the mechanical system described by Figure 1. Define the states $x_1 = \phi$, $x_2 = \dot{\phi}$, $x_3 = D$, and $x_4 = \dot{D}$. Analysis of Figure 1 (and application of some reasonable approximations) reveals that

$$
\dot{x}_1(t) = x_2(t) \\
\dot{x}_2(t) = \frac{g}{L} \sin x_1(t) - \frac{1}{LM} (-fx_4 + u(t)) \cos x_1(t) \\
\dot{x}_3(t) = x_4(t) \\
\dot{x}_4(t) = -\frac{f}{M}x_4(t) + \frac{1}{M}u(t)
$$

Suppose also that the output of this system is $y(t) = \tan x_1$.

(a) Observe that $x_1(t) = x_2(t) = x_3(t) = x_4(t) = u(t) = 0$ is a solution to this set of differential equations. Linearize this system around this solution and find $A, B, C, D$ such that $\dot{x}(t) = Ax(t) + Bu(t)$ and $y(t) = Cx(t) + Du(t)$.

(b) Observe that $x_1(t) = \pi$ and $x_2(t) = x_3(t) = x_4(t) = u(t) = 0$ is another solution to this set of differential equations. Linearize this system around this solution and find $A, B, C, D$ such that $\dot{x}(t) = Ax(t) + Bu(t)$ and $y(t) = Cx(t) + Du(t)$.

5. 3 pts. For each of the following, find a solution for $x$. If a solution does not exist then show why. If the solution is not unique then mathematically describe the set of all possible solutions. Justify your answers.

(a) 

$$
\begin{bmatrix}
1 & 2 & 3 \\
2 & 4 & 6 \\
3 & 6 & 10 \\
4 & 7 & 13
\end{bmatrix}
\begin{bmatrix}
x
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
1 \\
3
\end{bmatrix}
$$

(b) 

$$
\begin{bmatrix}
1 & 2 & 3 \\
2 & 4 & 6 \\
3 & 6 & 10 \\
4 & 7 & 13
\end{bmatrix}
\begin{bmatrix}
x
\end{bmatrix} =
\begin{bmatrix}
1 \\
1 \\
1 \\
3
\end{bmatrix}
$$

(c) 

$$
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 4 & 6 & 7 \\
3 & 6 & 10 & 13
\end{bmatrix}
\begin{bmatrix}
x
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
$$
6. 3 pts. Given arbitrary $P \in \mathbb{R}^{m \times n}$ and $Q \in \mathbb{R}^{n \times k}$. Show that, if the columns of $PQ$ are linearly independent, then so are the columns of $Q$. Give an example to show that the converse of this statement is not true, in general. Hint: Write $Q = [q_1, \ldots, q_k]$. If the columns of $Q$ are linearly dependent then there exists a set of $k$ scalars $\{\alpha_i\}_{i=1}^k$ such that $\alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_k q_k = 0$.

7. 5 pts. Given the following discrete time, LTI, state-space system description,

$$
\begin{align*}
\mathbf{x}[k+1] &= \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}[k] + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u[k] \\
y[k] &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \mathbf{x}[k] + u[k]
\end{align*}
$$

(a) Find a different state-space realization of this system that has the same impulse response as this system.

(b) For $k \geq 0$, explicitly compute the zero-input response of the system for the following cases:

$$
\begin{align*}
\mathbf{x}(0) &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \\
\mathbf{x}(0) &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \\
\mathbf{x}(0) &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},
\end{align*}
$$

(c) Use your result from part (a) to derive a general expression for the zero-input response of the system when the initial state $\mathbf{x}(0) = [\gamma_1, \gamma_2, \gamma_3]^T$. 