ECE531 Screencast 1.4: Minimum Variance Unbiased Estimators

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Minimum Variance Unbiased Estimators

**Definition**

A minimum-variance unbiased estimator $\hat{\theta}_{\text{mvu}}(y)$ is an unbiased estimator satisfying

$$
\hat{\theta}_{\text{mvu}}(y) = \arg \min_{\hat{\theta}(y) \in \Omega} R_\theta(\hat{\theta}(y))
$$

for all $\theta \in \Lambda$ where $\Omega$ is the set of all unbiased estimators.

**Remarks:**

- Finding an MVU estimator is a multi-objective optimization problem. You have to find **one** estimator to minimize the variance at all $\theta \in \Lambda$.
- The estimator cannot be a function of $\theta$.
- MVU estimators do not always exist (see Example 2.3 in Kay I).
- We will see, however, that lots of problems do yield MVU estimators.
Example: Estimating a Constant in White Gaussian Noise

Suppose we have random observations given by

\[ Y_k = \theta + W_k \quad k = 0, \ldots, n - 1 \]

where \( W_k \sim \text{i.i.d.} \mathcal{N}(0, \sigma^2) \) with \( \theta \in \mathbb{R} \).

Suppose we go with an estimator that performs the sample mean:

\[ \hat{\theta}(y) = \frac{1}{n} \sum_{k=0}^{n-1} y_k \]

- Is this estimator unbiased? Yes (easy to check).
- Is this estimator MVU? The variance of the estimator can be calculated as \( \text{var}_\theta \left[ \hat{\theta}(Y) \right] = \frac{\sigma^2}{n} \). But answering the question as to whether this estimator is MVU or not will require more work.
Example: Estimating Mean and Variance

Suppose we have random observations given by

\[ Y_k = \theta_1 + W_k \quad k = 0, \ldots, n - 1 \]

where \( W_k \) i.i.d. \( \sim \mathcal{N}(0, \theta_2) \). Note that both \( \theta_1 \) and \( \theta_2 > 0 \) are unknown. This is called a vector parameter estimation problem.

We already know an unbiased estimator for \( \theta_1 \): the sample mean

\[ \hat{\theta}_1(y) = \frac{1}{n} \sum_{k=0}^{n-1} y_k \]

This estimator is still valid in this case because the sample mean does not depend on any unknown parameters.

How about this estimator for \( \theta_2 \) (sample variance):

\[ \hat{\theta}_2(y) = \frac{1}{n} \sum_{k=0}^{n-1} (y_k - \hat{\theta}_1(y))^2 \]

- Is this estimator unbiased?
- Is this estimator MVU?
Example: Estimating Mean and Variance

To check if the estimator for $\theta_2$ is unbiased, we can calculate the mean:

$$E_\theta[\hat{\theta}_2(Y)] = E_\theta \left[ \frac{1}{n} \sum_{\ell=0}^{n-1} \left( Y_\ell - \hat{\theta}_1(Y) \right)^2 \right]$$

$$= \frac{1}{n} \sum_{\ell=0}^{n-1} E_\theta \left[ \left( Y_\ell - \hat{\theta}_1(Y) \right)^2 \right]$$

$$= \frac{1}{n} \sum_{\ell=0}^{n-1} E_\theta \left[ \left( Y_\ell - \left( \frac{1}{n} \sum_{k=0}^{n-1} Y_k \right) \right)^2 \right]$$

$$= \frac{1}{n} \sum_{\ell=0}^{n-1} E_\theta \left[ \left( \theta_1 + W_\ell - \left( \frac{1}{n} \sum_{k=0}^{n-1} (\theta_1 + W_k) \right) \right)^2 \right]$$

$$= \frac{1}{n^3} \sum_{\ell=0}^{n-1} E_\theta \left[ \left( (n-1)W_\ell - \sum_{k=0}^{n-1} W_k \right)^2 \right]$$

$$= \frac{1}{n^3} \sum_{\ell=0}^{n-1} \left[ (n-1)^2 + n - 1 \right] \theta_2 = \frac{n-1}{n} \theta_2$$
Example: Estimating Mean and Variance

Since

$$E_\theta[\hat{\theta}_2(Y)] = \frac{n-1}{n} \theta_2 \neq \theta_2$$

this estimator is biased.

We can use this result to make an unbiased estimator for $\theta_2$, however:

$$\hat{\theta}_2(y) = \frac{1}{n-1} \sum_{k=0}^{n-1} (y_k - \hat{\theta}_1(y))^2$$

You can confirm this estimator is unbiased by computing the mean $E_\theta[\hat{\theta}_2(Y)]$ as we did before.
Finding MVU Estimators

There is no “plug-and-chug” method that you can always follow to find an MVU estimator. We will cover two common approaches that work in many cases:

1. The Cramer-Rao lower bound (Kay I: Chapters 3-4)
   ▶ Guess at a good estimator and check if the variance achieves the theoretical minimum (CRLB).
   ▶ Take advantage of special cases, e.g. linear model

2. The Rao-Blackwell-Lehmann-Sheffe (RBLS) theorem (Kay I: Chapter 5)
   ▶ Finding a “complete sufficient statistic” for the observations.
   ▶ Performing a conditional expectation to get the MVU estimator.