Problem Statement

2.9 This problem illustrates what happens to an unbiased estimator when it undergoes a nonlinear transformation. In example 2.1, if we choose to estimate the unknown parameter $\theta = A^2$ by

$$\hat{\theta} = \left( \frac{1}{N} \sum_{n=0}^{N-1} x[n] \right)^2$$

can we say that the estimator is unbiased? What happens as $N \to \infty$?

Example 2.1 is about estimating a DC level $A$ in white Gaussian noise. We know the sample mean estimator is an unbiased estimator of $A$. This problem is about estimating $A^2$, however. The question is can we just square the sample mean estimate to get an unbiased estimate of $A^2$?
To answer the question about bias, recall that $X[n] = A + W[n]$ with $W[n] \sim \mathcal{N}(0, \sigma^2)$ and let’s compute the mean of our estimator:

$$E_\theta(\hat{\theta}(X)) = E_\theta \left( \left( \frac{1}{N} \sum_{n=0}^{N-1} X[n] \right)^2 \right)$$

$$= \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} E_\theta(X[n]X[m])$$

$$= \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} E_\theta((A + W[n])(A + W[m]))$$

$$= \frac{1}{N^2} \left[ N(A^2 + \sigma^2) + N(N-1)A^2 \right]$$

$$= A^2 + \frac{\sigma^2}{N}$$

so this is clearly biased if $\sigma^2 > 0$ and $N < \infty$. 
Remarks

Using this result, we could form an unbiased estimator

\[
\hat{\theta} = \left( \frac{1}{N} \sum_{n=0}^{N-1} x[n] \right)^2 - \frac{\sigma^2}{N}.
\]

This estimator is valid because \( \sigma^2 \) and \( N \) are known.

Finally, what happens as \( N \to \infty \)? We see from the previous result that the bias vanishes. We call such an estimator “asymptotically unbiased”. Even though it is biased for finite \( N \), the bias vanishes as \( N \to \infty \).