ECE531 Screencast 10.4: Minimum Error Probability Binary Signal Detection

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Minimum Probability of Error: Bayes Detection with UCA

Suppose we have a binary additive Gaussian noise detection problem with $x \in \{ s_0, s_1 \}$ and a given prior $\pi_0$.

The probability of making an incorrect decision is (Bayes risk with UCA)

$$
P_e = \pi_0 \text{Prob}(\text{decide } \mathcal{H}_1 \mid x = s_0) + (1 - \pi_0) \text{Prob}(\text{decide } \mathcal{H}_0 \mid x = s_1)
$$

$$
= \pi_0 \text{Prob} \left( \bar{s}^\top \bar{Y} \geq v' \mid x = s_0 \right) + (1 - \pi_0) \text{Prob} \left( \bar{s}^\top \bar{Y} < v' \mid x = s_1 \right)
$$

where $v'$ is the decision threshold. For $\mu_j := (s_1 - s_0)^\top \bar{s}_j$, we can write

$$
P_e = \pi_0 Q \left( \frac{v' - \mu_0}{\| \bar{s} \|} \right) + (1 - \pi_0) Q \left( \frac{\mu_1 - v'}{\| \bar{s} \|} \right)
$$

and we know the decision threshold $v'$ that minimizes $P_e$ is

$$
v' = \ln \frac{\pi_0}{1 - \pi_0} + \frac{1}{2} (\| \bar{s}_1 \|^2 - \| \bar{s}_0 \|^2)
$$
Uniform Prior and Optimum Detection Threshold

Suppose that we have a uniform prior \( \pi_0 = \pi_1 = \frac{1}{2} \) and that the optimum threshold for this prior, i.e. \( v' = \frac{1}{2}(||\bar{s}_1||^2 - ||\bar{s}_0||^2) \), is used by the detector. Note that

\[
v' = \frac{1}{2}(||\bar{s}_1||^2 - ||\bar{s}_0||^2) = \frac{(\bar{s}_1 - \bar{s}_0)^\top \bar{s}_0 + (\bar{s}_1 - \bar{s}_0)^\top \bar{s}_1}{2} = \frac{\mu_0 + \mu_1}{2}
\]

Hence, the probability of error can be written as

\[
P_e = \frac{1}{2}Q\left(\frac{\mu_1 - \mu_0}{2||\bar{s}||}\right) + \frac{1}{2}Q\left(\frac{\mu_1 - \mu_0}{2||\bar{s}||}\right) = Q\left(\frac{\mu_1 - \mu_0}{2||\bar{s}||}\right)
\]

We can also write

\[
\mu_1 - \mu_0 = (\bar{s}_1 - \bar{s}_0)^\top \bar{s}_1 - (\bar{s}_1 - \bar{s}_0)^\top \bar{s}_0 = ||\bar{s}_0||^2 + ||\bar{s}_1||^2 - 2\bar{s}_0^\top \bar{s}_1 = ||\bar{s}||^2
\]

Hence

\[
P_e = Q\left(\frac{||\bar{s}||}{2}\right)
\]
Optimum Signal Design for Minimum Error Probability

In communication problems, we often can choose our signal alphabet. Since \( Q(x) \) is monotonically decreasing in \( x \), we should choose \( s_0 \) and \( s_1 \) to maximize \( \|\vec{s}\| \). To get an useful result, we require the power of each signal \( s_0 \) and \( s_1 \) to be upper bounded by \( B \), i.e.

\[
\frac{1}{n} \| s_j \|^2 \leq B \quad \Rightarrow \quad \frac{1}{n} \| s_0 - s_1 \|^2 \leq 4B
\]

Recall \( \vec{s} = (Ss_1 - Ss_0) \) with \( \Sigma^{-1} = S^\top S \). We can write

\[
\| \vec{s} \|^2 = (Ss_1 - Ss_0)^\top (Ss_1 - Ss_0) = (s_1 - s_0)^\top \Sigma^{-1} (s_1 - s_0)
\]

Let \( \lambda_{max}(\Sigma^{-1}) \) be the largest eigenvalue of the positive definite matrix \( \Sigma^{-1} \) and let \( \nu \) be the eigenvector associated with this eigenvalue. Then

\[
\| \vec{s} \|^2 \leq \lambda_{max}(\Sigma^{-1}) \| s_1 - s_0 \|^2
\]

with equality only if \( s_1 - s_0 = \alpha \nu \) for some scalar \( \alpha \). Hence

\[
(s_1 - s_0)^\top \Sigma^{-1} (s_1 - s_0) = \| \vec{s} \|^2 \leq 4nB \lambda_{max}(\Sigma^{-1}).
\]
The probability of error is minimized when \( s_1 - s_0 \) is aligned with the eigenvector of \( \Sigma^{-1} = 1/\lambda_{\text{min}}(\Sigma) \) corresponding to the maximum eigenvalue \( \lambda_{\text{max}}(\Sigma^{-1}) \) and arranged antipodally on the sphere of radius \( \sqrt{nB} \).
The probability of error is minimized when $s_1 - s_0$ is aligned with the eigenvector of $\Sigma$ corresponding to the smallest eigenvalue $\lambda_{\text{min}}(\Sigma)$, i.e. $s_1 - s_0$ is aligned in the direction of least noise variance.

The minimum achievable error probability with optimum signaling (and equiprobable signals) is then

$$P_e^* = Q\left(\frac{||\bar{s}||}{2}\right) = Q\left(\frac{\sqrt{4nB\lambda_{\text{max}}(\Sigma^{-1})}}{2}\right) = Q\left(\frac{\sqrt{nB}}{\sigma_{\text{min}}}\right)$$

where $\sigma_{\text{min}}^2 = \lambda_{\text{min}}(\Sigma)$. 