ECE531 Screencast 11.2: Bayesian Composite Hypothesis Testing

D. Richard Brown III

Worcester Polytechnic Institute
The Bayesian Approach

The Bayesian approach is to

- assume a prior distribution $\pi$ on the states and
- form a single metric (the Bayes risk $r(\rho, \pi)$) that we minimize.

From state $x$, we have the conditional risk

$$R_x(\rho) = \int_y \rho^\top(y) C(x) p_x(y) \, dy$$

and the Bayes risk is

$$r(\rho, \pi) = \sum_{j=0}^{N-1} \pi_j R_j(\rho) \quad \text{(finite number of states)}$$

$$r(\rho, \pi) = \sum_{x \in \mathcal{X}} \pi_x R_x(\rho) \quad \text{(countably infinite number of states)}$$

$$r(\rho, \pi) = \int_{x \in \mathcal{X}} \pi(x) R_x(\rho) \quad \text{(uncountably infinite number of states)}$$
Bayes Risk and Bayesian Decision Rule

The Bayes risk of decision rule $\rho$ can be written as

$$r(\rho, \pi) = \int_X \pi(x) R_x(\rho) \, dx$$

$$= \int_X \pi(x) \int_Y \rho^\top(y) C(x) p_x(y) \, dy \, dx$$

$$= \int_Y \rho^\top(y) \left( \int_X C(x) \pi(x) p_x(y) \, dx \right) \, dy$$

$$g(y,\pi) = [g_0(y,\pi), \ldots, g_{M-1}(y,\pi)]^\top$$

The Bayesian decision rule $\rho(y) = \delta^{B\pi}(y)$ is then

$$\delta^{B\pi}(y) = \arg\min_{i \in \{0, \ldots, M-1\}} \sum_{x \in X} C_i(x) \pi_x p_x(y)$$
Bayes Decision Rules for Composite HT Problems

It doesn't matter whether we have a finite, countably infinite, or uncountably infinite number of states, we can always find a deterministic Bayes decision rule as

$$\delta^{B\pi}(y) = \arg\min_{i \in \{0, \ldots, M-1\}} g_i(y, \pi)$$

where

$$g_i(y, \pi) = \begin{cases} \sum_{j=0}^{N-1} C_{i,j} \pi_j p_j(y) & \text{when } \mathcal{X} = \{x_0, \ldots, x_{N-1}\} \text{ is finite} \\ \sum_{x \in \mathcal{X}} C_i(x) \pi_x p_x(y) & \text{when } \mathcal{X} \text{ is countably infinite} \\ \int_{\mathcal{X}} C_i(x) \pi(x) p_x(y) \, dx & \text{when } \mathcal{X} \text{ is uncountably infinite} \end{cases}$$

is the usual “commodity cost” associated with hypothesis $\mathcal{H}_i$.

When we have only two hypotheses, we just have to compare $g_0(y, \pi)$ with $g_1(y, \pi)$ at each value of $y$. We decide $\mathcal{H}_1$ when $g_1(y, \pi) < g_0(y, \pi)$, otherwise we decide $\mathcal{H}_0$. 
Example (part 1 of 3)

Consider a scenario with $N = 3$ states and $M = 2$ hypotheses. The states are given as

\begin{align*}
x_0 : y &\sim \mathcal{N}(0, 1) \\
x_1 : y &\sim \mathcal{N}(-2, 1) \\
x_2 : y &\sim \mathcal{N}(2, 1)
\end{align*}

The hypothesis are given as $\mathcal{H}_0 = \{x_0\}$ and $\mathcal{H}_1 = \{x_1, x_2\}$.

Further assume $\pi_0 = \pi_1 = \pi_2 = \frac{1}{3}$ and the UCA.

Since we have a finite number of states, we can use a cost matrix. The cost matrix in this case can be written as

\[
C = \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 0
\end{bmatrix}
\]
Example (part 2 of 3)

To find the Bayesian decision rule, we can compute the two “commodity costs” as

\[
g_0(y, \pi) = \sum_{j=0}^{2} C_{0,j} \pi_j p_j(y) = \frac{1}{3} (p_1(y) + p_2(y))
\]

\[
g_1(y, \pi) = \sum_{j=0}^{2} C_{1,j} \pi_j p_j(y) = \frac{1}{3} p_0(y)
\]

We decide \( \mathcal{H}_1 \) if and only if \( g_1(y, \pi) < g_0(y, \pi) \) which we can write as

\[
g_1(y, \pi) < g_0(y, \pi) \iff \frac{g_0(y, \pi)}{g_1(y, \pi)} > 1
\]

\[
\iff \frac{p_1(y) + p_2(y)}{p_0(y)} > 1
\]

\[
\iff \exp(2y - 2) + \exp(-2y - 2) > 1
\]
Example (part 3 of 3)

\[ y \rightarrow \text{decision statistic } = \exp(2y-2) + \exp(-2y-2) \]

Decision statistic and decision threshold.