ECE531 Screencast 12.5: Detection of a Known Signal with Unknown Parameters in a Linear Model

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Linear Model

Suppose we have observations given by

\[ y = Hx + w \]

where \( H \in \mathbb{R}^{n \times p} \) is known, \( x \in \mathbb{R}^{p \times 1} \) is a vector of parameters/states, at least some of which are unknown, and \( W \in \mathbb{R}^{n \times 1} \) with \( W \sim \mathcal{N}(0, \sigma^2 I) \) is AWGN with \( \sigma^2 \) known.

We want to decide between

\[ \mathcal{H}_0 : x = 0 \]
\[ \mathcal{H}_1 : x \neq 0 \]

Many problems fit into this linear model, even some that aren’t obvious.
Example: Sinusoidal Signal with Unknown Amp. and Phase

Suppose the signal we are trying to detect is

\[ s_k = a \cos(\omega k + \phi) \]

for \( k = 0, \ldots, n - 1 \) with unknown \( a \) and \( \phi \). We can write

\[ s_k = \alpha_1 \cos(\omega k) + \alpha_2 \sin(\omega k) \]

with \( \alpha_1^2 + \alpha_2^2 = a^2 \). Letting the unknown parameters \( x = [\alpha_1, \alpha_2]^T \), we have

\[ y = Hx + w \]

with

\[ H = \begin{bmatrix}
1 & 0 \\
\cos(\omega_0) & \sin(\omega_0) \\
\vdots & \vdots \\
\cos(\omega_0(n-1)) & \sin(\omega_0(n-1))
\end{bmatrix} \]
GLRT for Linear Model

Since the MLE for the linear model with AWGN is simply

\[ \hat{x} = (H^\top H)^{-1} H^\top y \]

the GLRT decision rule can be written as follows. We decide \( \mathcal{H}_1 \) if

\[
\frac{p_1(y; \hat{x})}{p_0(y)} > v
\]

\[\Leftrightarrow \quad \exp \left( -\frac{(y-H\hat{x})^\top (y-H\hat{x})}{2\sigma^2} \right) > v \]

\[\Leftrightarrow \quad 2\hat{x}^\top H^\top y - \hat{x} H^\top H \hat{x} > v' \]

\[\Leftrightarrow \quad 2y^\top H(H^\top H)^{-1} H^\top y - y^\top H(H^\top H)^{-1} H^\top H(H^\top H)^{-1} H^\top y > v' \]

\[\Leftrightarrow \quad y^\top H(H^\top H)^{-1} H^\top y > v' \]

with \( v' \) chosen to satisfy \( P_{fp} \leq \alpha \). The decision statistic is \( \chi^2 \) distributed, so we can get exact expressions for \( P_{fp} \) and \( P_D \).
Bayesian Detection for Linear Model with Gaussian Prior

In the Bayesian case with $Y = Hx + W$, we assume a Gaussian prior on $x$ such that

$$\pi(x) = \pi_0 \delta(x) + (1 - \pi_0) \frac{1}{(2\pi)^{n/2} |\Sigma_x|^{1/2}} \exp \left( -\frac{(x - \mu_x)\Sigma_x^{-1}(x - \mu_x)}{2} \right).$$

If we let $s = Hx$ then $s$ is also Gaussian with $\mu_s = H\mu_x$ and $\Sigma_s = H\Sigma_x H^\top$. We assume $\mu_s$ and $\Sigma_s$ are known and $s$ is independent of the noise $W \sim \mathcal{N}(0, \sigma^2 I)$.

We want to decide between

$$\mathcal{H}_0 : x = 0 \iff s = 0$$
$$\mathcal{H}_1 : x \neq 0 \iff s \neq 0$$

which is equivalent to

$$\mathcal{H}_0 : Y \sim \mathcal{N}(0, \sigma^2 I)$$
$$\mathcal{H}_1 : Y \sim \mathcal{N}(\mu_s, \Sigma_s + \sigma^2 I)$$

This is actually a simple binary hypothesis testing problem!
Bayesian Detection for Linear Model with Gaussian Prior

Since this problem is simple and binary, we know we decide $H_1$ when

$$\frac{p_1(y)}{p_0(y)} > \frac{\pi_0}{1 - \pi_0}$$

$$\iff \frac{1}{(2\pi)^{n/2} |\Sigma_s + \sigma^2 I|^{1/2}} \exp \left( -\frac{(y - \mu_s)^\top (\Sigma_s + \sigma^2 I)^{-1} (y - \mu_s)}{2} \right) > \frac{\pi_0}{1 - \pi_0}$$

$$\iff \frac{|\sigma^2 I|^{1/2}}{|\Sigma_s + \sigma^2 I|^{1/2}} \exp \left( -\frac{(y - \mu_s)^\top (\Sigma_s + \sigma^2 I)^{-1} (y - \mu_s)}{2} + \frac{y^\top y}{2\sigma^2} \right) > \frac{\pi_0}{1 - \pi_0}$$

$$\iff \frac{y^\top y}{\sigma^2} - (y - \mu_s)^\top (\Sigma_s + \sigma^2 I)^{-1} (y - \mu_s) > 2 \ln \left( \frac{\pi_0 |\Sigma_s + \sigma^2 I|^{1/2}}{(1 - \pi_0)|\sigma^2 I|^{1/2}} \right)$$

$$\iff y^\top Ay + \mu_s^\top By > 2 \ln \left( \frac{\pi_0 |\Sigma_s + \sigma^2 I|^{1/2}}{(1 - \pi_0)|\sigma^2 I|^{1/2}} \right) + \mu_s^\top (\Sigma_s + \sigma^2 I)^{-1} \mu_s$$

with

$$A = \frac{1}{\sigma^2} I - (\Sigma_s + \sigma^2 I)^{-1} \quad \text{and} \quad B = 2(\Sigma_s + \sigma^2 I)^{-1}$$