ECE531 Screencast 13.4: Generalized Likelihood Ratio Test Detection with Unknown AWGN Noise Variance

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GLRT Decision Rules for Composite HT Problems

Given a composite hypothesis testing problem with unknown state $x$. Recall the main idea of the GLRT:

- get an observation $y$
- estimate the most likely value of $x$ under $\mathcal{H}_0$ (call this $\hat{x}_0$)
- estimate the most likely value of $x$ under $\mathcal{H}_1$ (call this $\hat{x}_1$)

and then form the GLRT

$$\frac{p_1(y; \hat{x}_1)}{p_0(y; \hat{x}_0)} > v$$

where $v$ is selected to satisfy the false positive probability constraint.
GLRT: Known Signal in AWGN with Unknown Variance

We have the binary hypothesis testing problem

$$\mathcal{H}_0 : Y = W$$
$$\mathcal{H}_1 : Y = s + W$$

with $s \in \mathbb{R}^n$ known and $W \sim \mathcal{N}(0, \sigma^2 I)$ with $\sigma^2$ unknown.

The MLE of $\sigma^2$ under $\mathcal{H}_0$ is $\hat{\sigma}_0^2 = \frac{1}{n} y^\top y$ and the MLE of $\sigma^2$ under $\mathcal{H}_1$ is $\hat{\sigma}_1^2 = \frac{1}{n} (y - s)^\top (y - s)$.

The GLRT (with $x = \sigma^2$) is then

$$\frac{\max_{x \in \mathcal{X} \backslash \mathcal{X}_0} p_1(y; x)}{\max_{x \in \mathcal{X}_0} p_0(y; x)} = \frac{1}{(2\pi \hat{\sigma}_1^2)^{n/2}} \exp \left( -\frac{(y-s)^\top (y-s)}{2\hat{\sigma}_1^2} \right)$$

Unfortunately, the statistics of $Z = \frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}$ depend on $\sigma^2$ under $\mathcal{H}_0$, so the threshold $v$ will depend on $\sigma^2$ to satisfy a false positive probability constraint. This detector is not CFAR.
GLRT: Known Signal in AWGN with Unknown Variance

Histogram of GLRT test statistic (under H0)

- $\sigma^2 = 16$
- $\sigma^2 = 100$
Now consider the case when we wish to detect a known signal with unknown parameters in AWGN with unknown variance. Suppose the observations $y \in \mathbb{R}^n$ are received from the classical linear model

$$y = Hx + w$$

with $x \in \mathbb{R}^{p \times 1}$ containing the unknown signal parameters and $W \sim \mathcal{N}(0, \sigma^2 I)$ with $\sigma^2$ unknown.

The hypotheses are given as

$$\mathcal{H}_0 : x = 0 \text{ and } \sigma^2 > 0$$

$$\mathcal{H}_1 : x \neq 0 \text{ and } \sigma^2 > 0$$
Under the conditions of the previous slide, the GLRT decides $\mathcal{H}_1$ if

$$T(y) = \frac{n - p}{p} \frac{\hat{x}_1^\top H^\top H \hat{x}_1}{y^\top (I - H(H^\top H)^{-1} H^\top) y} > v$$

where $\hat{x}_1 = (H^\top H)^{-1} H^\top y$ is the MLE of $x$ under $\mathcal{H}_1$. The decision statistic in this case is CFAR and

$$P_{fp} = Q_U(v)$$
$$P_D = Q_V(v)$$

where $Q_Z(x) = \int_x^\infty p_Z(t) \, dt$ is the tail probability of the random variable $Z$ and

- $U \sim F_{p,n-p}$ denotes the $F$ distribution with $p$ numerator degrees of freedom and $n - p$ denominator degrees of freedom
- $V \sim F'_{p,n-p}(\lambda)$ denotes the non-central $F$ distribution with $p$ numerator degrees of freedom, $n - p$ denominator degrees of freedom, and non-centrality parameter

$$\lambda = \frac{x^\top H^\top H x}{\sigma^2}$$

with $x$ denoting the true value of the unknown signal parameters.
GLRT for Classical Linear Model in AWGN with Unk. $\sigma^2$

To provide some interpretation, let’s rewrite the decision statistic

$$T(y) = \frac{n - p}{p} \left( \frac{\hat{x}_1^\top H^\top H \hat{x}_1}{y^\top (I - H(H^\top H)^{-1}H^\top) y} \right)$$

$$\quad = \frac{n - p}{p} \left( \frac{((H^\top H)^{-1}H^\top y)^\top H^\top H(H^\top H)^{-1}H^\top y}{y^\top (I - H(H^\top H)^{-1}H^\top) y} \right)$$

$$\quad = \frac{n - p}{p} \left( \frac{y^\top H(H^\top H)^{-1}H^\top y}{y^\top (I - H(H^\top H)^{-1}H^\top) y} \right)$$

$$\quad = \frac{n - p}{p} \left( \frac{y^\top P y}{y^\top (I - P) y} \right) = \frac{n - p}{p} \left( \frac{\|P y\|^2}{\|(I - P) y\|^2} \right)$$

where $P$ is an orthogonal projection matrix onto the subspace of $\mathbb{R}^n$ spanned by the columns of $H$ (the signal subspace) and $I - P$ is another orthogonal projection matrix onto the subspace of $\mathbb{R}^n$ orthogonal to the subspace spanned by the range of $H$ (the noise subspace).