ECE531 Screencast 2.4: Fisher Information for Vector Parameters

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Vector Parameter Estimation Problems

In many problems, we have more than one parameter that we would like to estimate. For example,

\[ Y_k = a \cos(\omega k + \phi) + W_k \] for \( k = 0, 1, \ldots, n - 1 \)

where \( a > 0, \phi \in (-\pi, \pi), \) and \( \omega \in (0, \pi) \) are all non-random parameters and \( W_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2) \). In this problem \( \theta = [a, \phi, \omega] \).

![Graph showing a plot of a cosine function with random noise](image-url)
Fisher Information Matrix

Recall, in the scalar parameter case, the Fisher information was motivated by a computation of the mean squared relative slope of the likelihood function:

$$ I(\theta) := E \left[ \left( \frac{\partial}{\partial \theta} \frac{p_Y(y ; \theta)}{p_Y(y ; \theta)} \right)^2 \right] = \int_Y \left( \frac{\partial}{\partial \theta} \ln p_Y(Y ; \theta) \right)^2 p_Y(y ; \theta) \, dy $$

In multiparameter problems, we are now concerned with the relative steepness of the likelihood function with respect to each of the parameters. A natural choice (assuming that all of the required derivatives exist) would be

$$ I(\theta) = E \left[ (\nabla_\theta \ln p_Y(Y ; \theta)) (\nabla_\theta \ln p_Y(Y ; \theta))^\top \right] \in \mathbb{R}^{m \times m} $$

where $\nabla_x$ is the gradient operator defined as

$$ \nabla_x f(x) := \left[ \frac{\partial}{\partial x_0} f(x), \ldots, \frac{\partial}{\partial x_{m-1}} f(x) \right]^\top. $$
Fisher Information Matrix

Let \( p := p_Y(Y; \theta) \). The Fisher information matrix is then

\[
I(\theta) = \begin{bmatrix}
E \left[ \frac{\partial}{\partial \theta_0} \ln p \cdot \frac{\partial}{\partial \theta_0} \ln p \right] & \cdots & E \left[ \frac{\partial}{\partial \theta_0} \ln p \cdot \frac{\partial}{\partial \theta_{m-1}} \ln p \right] \\
E \left[ \frac{\partial}{\partial \theta_1} \ln p \cdot \frac{\partial}{\partial \theta_0} \ln p \right] & \cdots & E \left[ \frac{\partial}{\partial \theta_1} \ln p \cdot \frac{\partial}{\partial \theta_{m-1}} \ln p \right] \\
\vdots & \ddots & \vdots \\
E \left[ \frac{\partial}{\partial \theta_{m-1}} \ln p \cdot \frac{\partial}{\partial \theta_0} \ln p \right] & \cdots & E \left[ \frac{\partial}{\partial \theta_{m-1}} \ln p \cdot \frac{\partial}{\partial \theta_{m-1}} \ln p \right]
\end{bmatrix}
\]

Note that the \( ij \)th element of the Fisher information matrix is given as

\[
I_{ij}(\theta) = E \left[ \frac{\partial}{\partial \theta_i} \ln p_Y(Y; \theta) \cdot \frac{\partial}{\partial \theta_j} \ln p_Y(Y; \theta) \right]
\]

hence we can say that \( I(\theta) \) is symmetric.
Fisher Information Matrix

When the second derivatives all exist, we can write

\[
\frac{\partial^2}{\partial \theta_i \partial \theta_j} \ln p_Y(y ; \theta) = \frac{\partial^2}{\partial \theta_i \partial \theta_j} p_Y(y ; \theta) \frac{\partial}{\partial \theta_i} p_Y(y ; \theta) \frac{\partial}{\partial \theta_j} p_Y(y ; \theta) - \frac{\partial}{\partial \theta_i} p_Y(y ; \theta) p_Y(y ; \theta) \frac{\partial}{\partial \theta_j} p_Y(y ; \theta) p_Y(y ; \theta)
\]

and, under the theorem’s assumptions, we can write

\[
E \left[ \frac{\partial^2}{\partial \theta_i \partial \theta_j} \ln p_Y(y ; \theta) \right] = -E \left[ \frac{\partial}{\partial \theta_i} \ln p_Y(y ; \theta) \cdot \frac{\partial}{\partial \theta_j} \ln p_Y(y ; \theta) \right] = -I_{ij}(\theta).
\]

Hence, we can say that

\[
I_{ij}(\theta) = -E \left[ \frac{\partial^2}{\partial \theta_i \partial \theta_j} \ln p_Y(y ; \theta) \right]
\]

This expression is often more convenient to compute than the former expression for \(I_{ij}(\theta)\).
Fisher Information Matrix

Under the conditions of the theorem

\[
E \left[ \frac{\partial}{\partial \theta_i} \ln p_Y \{Y \; ; \; \theta \} \right] = \int_Y \frac{\partial}{\partial \theta_i} p_Y (y \; ; \; \theta) \frac{p_Y (y \; ; \; \theta)}{p_Y (y \; ; \; \theta)} \, dy \\
= \frac{\partial}{\partial \theta_i} \int_Y p_Y (y \; ; \; \theta) \, dy = 0
\]

Hence

\[
I_{ij} (\theta) = \text{cov} \left\{ \frac{\partial}{\partial \theta_i} \ln p_Y \{Y \; ; \; \theta \}, \frac{\partial}{\partial \theta_j} \ln p_Y \{Y \; ; \; \theta \} \right\}.
\]

The Fisher information matrix \( I(\theta) \) is a covariance matrix and is invertible if the unknown parameters are linearly independent.
Example: Fisher Information Matrix of Signal in AWGN

Many problems require the estimation of unknown signal parameters in additive white Gaussian noise. The observations in this case can be modeled as

\[ Y_k = s_k(\theta) + W_k \text{ for } k = 0, 1, \ldots, n - 1 \]

where \( s_k(\theta) : \Lambda \rightarrow \mathbb{R} \) is a deterministic signal with an unknown vector parameter \( \theta \) and where \( W_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2) \). We assume \( \sigma^2 \) is a known parameter and that all of the regularity conditions are satisfied.

To compute the \( ij \)th element of the Fisher information matrix, we can write

\[
\frac{\partial^2}{\partial \theta_i \theta_j} \ln p_Y(Y ; \theta) = \frac{1}{\sigma^2} \sum_{k=0}^{n-1} \left\{ [Y_k - s_k(\theta)] \frac{\partial^2}{\partial \theta_i \theta_j} s_k(\theta) - \left( \frac{\partial}{\partial \theta_i} s_k(\theta) \right) \left( \frac{\partial}{\partial \theta_j} s_k(\theta) \right) \right\}
\]

Since \( E[Y_k] = s_k(\theta) \), the \( ij \)th element of the FIM can be written as

\[
I_{ij}(\theta) = -E \left[ \frac{\partial^2}{\partial \theta_i \theta_j} \ln p_Y(Y ; \theta) \right] = \frac{1}{\sigma^2} \sum_{k=0}^{n-1} \left( \frac{\partial}{\partial \theta_i} s_k(\theta) \right) \left( \frac{\partial}{\partial \theta_j} s_k(\theta) \right)
\]