ECE531 Screencast 2.6: Cramér-Rao Lower Bound Example

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Problem Statement

Suppose $Y$ is a scalar observation drawn from a parameterized Poisson distribution

$$p_Y(y; \theta) = \text{Prob}(Y = y) = \frac{\theta^y e^{-\theta}}{y!}$$

for $y = 0, 1, 2, \ldots$. We wish to estimate $\theta$. Find the Fisher information $I(\theta)$ and the CRLB for estimating the scalar parameter $\theta$. Can you find an MVU estimator that achieves the CRLB in this case?
Solution Step 1: Check the Conditions of the Theorem

To confirm the derivative exists for all $\theta \in \Lambda$ and all $y \in \mathcal{Y}$, we can compute

$$\frac{\partial}{\partial \theta} p_Y(y ; \theta) = \frac{y\theta y^{-1} e^{-\theta} - \theta e^{-\theta}}{y!}.$$

You can take another partial derivative with respect to $\theta$ to confirm the second derivative exists as well.

We also need to check that $E \left[ \frac{\partial}{\partial \theta} \ln p_Y(Y ; \theta) \right] = 0$. We can compute

$$E \left[ \frac{\partial}{\partial \theta} \ln p_Y(Y ; \theta) \right] = \int_{\mathcal{Y}} \frac{\partial}{\partial \theta} p_Y(y ; \theta) \, dy = \sum_{y=0}^{\infty} \frac{y\theta y^{-1} e^{-\theta} - \theta e^{-\theta}}{y!}$$

$$= e^{-\theta} \sum_{y=0}^{\infty} \frac{y\theta y^{-1} - \theta y}{y!}$$

$$= e^{-\theta} \sum_{y=1}^{\infty} \frac{\theta y^{-1}}{(y-1)!} - e^{-\theta} \sum_{y=0}^{\infty} \frac{\theta y}{y!}$$

$$= e^{-\theta} e^{\theta} - e^{-\theta} e^{\theta} = 0$$

So this verifies the conditions of the theorem are satisfied.
Solution Step 2: Compute the Fisher Information

With the theorem conditions all confirmed, we can compute

$$\frac{\partial}{\partial \theta} \ln p_Y(y; \theta) = \frac{\partial}{\partial \theta}(-\theta + y \ln \theta) = -1 + \frac{y}{\theta},$$

and

$$\frac{\partial^2}{\partial \theta^2} \ln p_Y(y; \theta) = -\frac{y}{\theta^2} < 0.$$

The Fisher information is given by

$$I(\theta) = -E \left\{ \frac{\partial^2}{\partial \theta^2} \ln p_Y(Y; \theta) \right\} = \frac{E\{Y\}}{\theta^2} = \frac{1}{\theta}.$$
Solution Step 3: Compute the CRLB and find MVU

From the Fisher information, CRLB is this case is simply

\[ \text{var}[\hat{\theta}(Y)] \geq \theta = \frac{1}{I(\theta)}. \]

To find an MVU estimator, let’s try

\[ \hat{\theta}(y) = y. \]

Since \( Y \) is Poisson, we have \( E\{\hat{\theta}(Y)\} = \theta \). So \( \hat{\theta}(y) \) is an unbiased estimator of \( \theta \).

Since \( Y \) is Poisson, we also have \( \text{var}\{\hat{\theta}(Y)\} = \theta \). So \( \hat{\theta}(y) \) achieves the CRLB and is MVU. You can also easily verify the CRLB attainability conditions here.