ECE531 Screencast 3.5: The RBLS Theorem

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Conceptual Model

Parameter space $\Lambda$, observation space $\mathcal{Y}$, complete sufficient statistic $T(y)$, unbiased estimator $\hat{g}(y)$, probabilistic model $p_Y(y; \theta)$.
Theorem

If \( \hat{g}(y) \) is any unbiased estimator of \( \theta \) and \( T \) is a sufficient statistic for the family \( \{ p_Y(y; \theta); \theta \in \Lambda \} \), then

\[
\tilde{g}[T(y)] := E[\hat{g}(Y) | T(Y) = T(y)]
\]

is

- A valid estimator of \( \theta \) (not a function of \( \theta \))
- An unbiased estimator of \( \theta \).
- Of lesser or equal variance than that of \( \hat{g}(y) \) for all \( \theta \in \Lambda \)

Additionally, if \( T \) is complete, then \( \tilde{g}[T(y)] \) is an MVU estimator of \( \theta \).
Example: Estimating a Constant in White Gaussian Noise

Suppose \( \theta \in \mathbb{R} \) and \( Y_k = \theta + W_k \) with \( W_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2) \) for \( k = 0, \ldots, n - 1 \). Then the joint distribution

\[
p_{Y}(y; \theta) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{k=0}^{n-1} (y_k - \theta)^2 \right\}.
\]

Let \( T(y) = \frac{1}{n} \sum_{k=0}^{n-1} y_k \). We know this is a complete sufficient statistic. Let's apply the RBLS theorem to find the MVU estimator...

- We could choose the unbiased estimator \( \hat{g}(y) = y_0 \).
- Now we need to compute

\[
\tilde{g}[T(y)] := \mathbb{E}[\hat{g}(Y) | T(Y) = T(y)] = \mathbb{E} \left[ Y_0 \mid \frac{1}{n} \sum Y_k = \frac{1}{n} \sum y_k \right]
\]

- To solve this, we can use a standard formula for the conditional expectation of a jointly Gaussian random variable...
Example (continued)

- Suppose $Z = [X, Y]^\top$ is jointly Gaussian distributed. We know that

  \[
  \mathbb{E}[X|Y = y] = \mathbb{E}[X] + \frac{\text{cov}(X, Y)}{\text{var}(Y)}(y - \mathbb{E}[Y]).
  \]

- In our problem, letting $\bar{Y} = \frac{1}{n} \sum_{k=0}^{n-1} Y_k$, we can use this result to write

  \[
  \mathbb{E} [Y_0 | \bar{Y} = t] = \mathbb{E}[Y_0] + \frac{\text{cov}(Y_0, \bar{Y})}{\text{var}(\bar{Y})} (t - \mathbb{E}[\bar{Y}])
  = \theta + \frac{\sigma^2}{\sigma^2} \left( \frac{1}{n} \sum y_k - \theta \right)
  = \frac{1}{n} \sum y_k
  \]

- Hence $\hat{\theta}_{\text{mvu}}(y) = \frac{1}{n} \sum y_k$ is an MVU estimator of $\theta$ (as expected).