ECE531 Screencast 8.1: Introduction to Detection

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Detection = Hypothesis Testing

Detection is the process of deciding among two or more possible underlying statistical situations (“hypotheses”) from noisy observations.

Examples of hypotheses:

- The coin is fair ($H_0$) or not fair ($H_1$).
- The approaching airplane is friendly ($H_0$) or unfriendly ($H_1$).
- This email is spam ($H_1$) or not spam ($H_0$).
- The medical treatment is effective ($H_1$) or ineffective ($H_0$).
- Communication receiver: Given a codebook with $M$ codewords, which codeword was sent ($\{H_0, \ldots, H_{M-1}\}$)?

More generally, we want to specify a decision rule that maps observations to decisions optimally in some sense.
States and Observations

- Let $x \in \mathcal{X} = \{x_0, \ldots, x_{N-1}\}$ denote the state, a hidden variable about which we wish to make an inference.
- The available observation is modeled as a random variable $Y$ taking on values in the set $\mathcal{Y}$.
- For each state $x \in \mathcal{X}$, we assume we have a probabilistic description of the random variable $Y$ when the state is $x$. The notation $p_x(y) = p_{Y|X}(y|X = x)$ (random $X$) or $p_x(y) = p_Y(y;x)$ (nonrandom $X$) is the density of the random variable $Y$ when the state is $x$. 

\[
\begin{array}{c}
x_0 & \rightarrow & y_0 \\
\rightarrow & \rightarrow & \rightarrow \\
x_1 & \rightarrow & y_0 \\
\rightarrow & \rightarrow & \rightarrow \\
\mathcal{X} & p_x(y) & \mathcal{Y} \\
\end{array}
\]

states | observations
Hypotheses and Decisions

▶ **Hypotheses** can be represented as a partition of \( \mathcal{X} \), denoted by \( \mathcal{H} = \{ \mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_{M-1} \} \) where

\[
\mathcal{H}_i \subseteq \mathcal{X} \\
\mathcal{H}_i \neq \emptyset \\
\mathcal{H}_i \cap \mathcal{H}_j = \emptyset \text{ for } i \neq j \text{ and} \\
\bigcup_i \mathcal{H}_i = \mathcal{X}
\]

▶ The set of possible **decisions** is then \( \mathcal{Z} = \{0, 1, \ldots, M - 1\} \) where decision \( i \) indicates the selection of hypothesis \( \mathcal{H}_i \). In other words, decision \( i \) is the decision that \( x \in \mathcal{H}_i \).

▶ If \( \mathcal{X} \) is finite, then we must have \( M \leq N \).
Types of Hypothesis Testing Problems

Recall $N = |\mathcal{X}|$ is the number of states (assume $\mathcal{X}$ is finite for now) and $M = |\mathcal{H}|$ is the number of hypotheses.

- If $M = 2$, then we have a **binary** hypothesis testing problem.
- If $M = N$, then we seek to decide the actual state. In this case we can take $\mathcal{H}_i = \{x_i\}$ and we have a **simple** hypothesis testing problem.
- If $M < N$ or $\mathcal{X}$ is infinite, then we have a **composite** hypothesis testing problem. At least one hypothesis contains more than one state.

Unlike a simple hypothesis with underlying distribution $p_x(y)$, a composite hypothesis does not completely specify the underlying distribution.

Our focus will be on simple hypothesis testing problems for now, but we will return to composite hypothesis testing in a few weeks.
Examples

We have a coin with $\text{Prob}(H) = q$ unknown.

1. Suppose $q$ can only take on two values: $q_0$ or $q_1$. What kind of hypothesis testing problem is this? **Binary, simple.**

2. Suppose $q$ can take on any value in the set $\{q_0, q_1, \ldots, q_{M-1}\}$ and we wish to determine which value it is. What kind of hypothesis testing problem is this? **$M$-ary, simple.**

3. Suppose $q$ can take on any value in the set $\{q_0, q_1, \ldots, q_{N-1}\}$ but only wish to know if it is $q_0$ or not (e.g. $q_0 = 0.5$ “is the coin fair?”). What kind of hypothesis testing problem is this? **Binary, composite $M = 2 < N$.**

4. Suppose $q$ can be any value in $[0, 1]$ and we want to determine this value. What kind of problem is this? **Estimation.**
Model Summary

H0

H1

$p_x(y)$

states

observations

hypotheses

decision rule