ECE531 Spring 2013 Quiz 2

Your Name: SOLUTION

Instructions: This quiz is worth a total of 100 points. The quiz is open book and open notes. You may also use a calculator. You may not use a computer, phone, or tablet. Please show your work on each problem and box/circle your final answers. Points may be deducted for a disorderly presentation of your solution.

1. 50 points. Suppose you have a coin that comes up heads with probability $0 \leq \theta \leq 1$. You toss a coin $n$ independent times and define the scalar observation $Y \in \{0, \ldots, n\}$ as the number of heads observed. You wish to estimate the unknown parameter $\theta$, i.e. the probability that the coin comes up heads, from this scalar observation.

We know that $Y$ is binomially distributed with

$$p_Y(y; \theta) = \text{Prob}(Y = y; \theta) = \binom{n}{y} \theta^y (1 - \theta)^{(n-y)}$$

for $y \in \{0, \ldots, n\}$ with $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. We also know $\text{E}[Y] = n\theta$ and $\text{var}[Y] = n\theta(1 - \theta)$.

Question: Is $\hat{\theta}(y) = \frac{y}{n}$ an MVU estimator?

2. 50 points. Suppose you receive three observations

$$Y_k = \cos(\omega k + \phi) + W_k$$

for $k = -1, 0, 1$ with $W_k \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2)$. Defining the unknown parameter as $\theta = [\omega, \phi]^\top$, compute the Fisher information matrix. Your final answer should be exact. Do not use any approximations.
1. To answer this question, we can calculate

\[
E \hat{\Theta}(Y) = \frac{\sum Y}{n} = \frac{n \hat{\Theta}}{n} = \hat{\Theta} \quad \text{unbiased}
\]

\[
\text{var} \{\hat{\Theta}(Y)\} = \frac{1}{n^2} \text{var} \{Y\} = \frac{n \Theta (1-\Theta)}{n^2} = \frac{\Theta (1-\Theta)}{n}
\]

Can we use the CRLB?

Check: \(E \left\{ \frac{\partial}{\partial \Theta} \ln \left( P_Y(Y; \Theta) \right) \right\} = 0 \)?

\[
\frac{\partial}{\partial \Theta} \ln P_Y(Y; \Theta) = \frac{\partial}{\partial \Theta} \left( \text{constant} + Y \ln(\Theta) + (n-Y) \ln(1-\Theta) \right)
\]

\[
= \frac{Y}{\Theta} - \frac{n-Y}{1-\Theta} = \frac{Y(1-\Theta) - (n-Y) \Theta}{\Theta (1-\Theta)}
\]

\[
E \left\{ \frac{Y-n\Theta}{\Theta (1-\Theta)} \right\} = \frac{E \{Y\} - n \Theta}{\Theta (1-\Theta)} = \frac{n \Theta - n \Theta}{\Theta (1-\Theta)} = 0
\]

So, ok, the condition is satisfied and we can use the CRLB.

\[
I(\Theta) = E \left\{ \left( \frac{\partial}{\partial \Theta} \ln P_Y(Y; \Theta) \right)^2 \right\} = E \left\{ \frac{(Y-n\Theta)^2}{\Theta (1-\Theta)} \right\}
\]

\[
= \frac{1}{\Theta^2 (1-\Theta)^2} E \left\{ (Y-n\Theta)^2 \right\}
\]

\[
= \frac{1}{\Theta^2 (1-\Theta)^2} \text{var} \{Y\} = \frac{n \Theta (1-\Theta)}{\Theta^2 (1-\Theta)^2} = \frac{n}{\Theta (1-\Theta)}
\]

Hence \( \text{var} \{\hat{\Theta}(Y)\} \geq \frac{\Theta (1-\Theta)}{n} \)

But our estimator achieves this bound, hence it is efficient and must be MVU.
2. Since we have AWGN in this problem, we can use (3.33).

\[ I_{ij} (\theta) = \frac{1}{\sigma^2} \sum_{k=-1}^{1} \left( \frac{\partial s[k;\theta]}{\partial \theta_i} \right) \left( \frac{\partial s[k;\theta]}{\partial \theta_j} \right) \]

\[ s[k;\theta] = \cos(\omega k + \phi) \quad \text{with} \quad \theta = \begin{bmatrix} \omega \\ \phi \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \]

\[ \frac{\partial s[k;\theta]}{\partial \theta_1} = -k \sin(\omega k + \phi) \]

\[ \frac{\partial s[k;\theta]}{\partial \theta_2} = -\sin(\omega k + \phi) \]

\[ I_{11} (\theta) = \frac{1}{\sigma^2} \sum_{k=-1}^{1} k^2 \sin^2(\omega k + \phi) = \frac{1}{\sigma^2} \left( \sin^2(\omega + \phi) + \sin^2(-\omega + \phi) \right) \]

\[ I_{12} (\theta) = \frac{1}{\sigma^2} \sum_{k=-1}^{1} k \sin^2(\omega k + \phi) = \frac{1}{\sigma^2} \left( -\sin^2(-\omega + \phi) + \sin^2(\omega + \phi) \right) \]

\[ I_{22} (\theta) = \frac{1}{\sigma^2} \sum_{k=-1}^{1} \sin^2(\omega k + \phi) = \frac{1}{\sigma^2} \left( \sin^2(\omega + \phi) + \sin^2(\phi) + \sin^2(-\omega + \phi) \right) \]

and \[ I(\theta) = \begin{bmatrix} I_{11}(\theta) & I_{12}(\theta) \\ I_{12}(\theta) & I_{22}(\theta) \end{bmatrix} \]

Can play some trig games here to get in terms of cosines, but this is sufficient.