ECE2311D10 Exam 4

Your Name: SOLUTION Your box #: ________

April 16, 2010

- Open book, open notes.
- Calculators permitted.
- Look over all of the questions before starting.
- Budget your time to allow yourself enough time to work on each question.
- Write neatly and show your work/reasoning!
- This exam is worth a total of 100 points.

problem 1	problem 2	problem 3	TOTAL
40 points	30 points	30 points	100 points

1. 40 points total. Suppose you have the aperiodic signal $x_1(t)$ given in Figure 1 below.

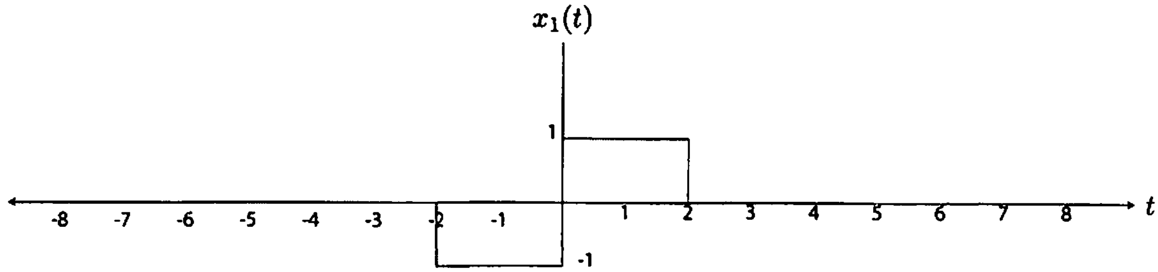


Figure 1: An aperiodic signal.

(a) 30 points. Compute the Fourier transform of $x_1(t)$, i.e. compute $X_1(\omega) = \mathcal{F}\{x_1(t)\}$. **Simplify your answer as much as possible using Euler's identity and other trigonometric identities.** You may find this particular trigonometric identity useful:

$$\sin^2(v) = \frac{1-\cos(2v)}{2}$$

$$X_1(\omega) = \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt = \int_{-2}^{0} (-1) e^{-j\omega t} dt + \int_{0}^{2} (1) e^{-j\omega t} dt$$

$$= \frac{-1}{-j\omega} e^{-j\omega t}\Big|_{-2}^{0} + \frac{1}{-j\omega} e^{-j\omega t}\Big|_{0}^{2}$$

$$= \frac{1}{j\omega}\left[1 - e^{j2\omega} - \left(e^{-j2\omega} - 1\right)\right]$$

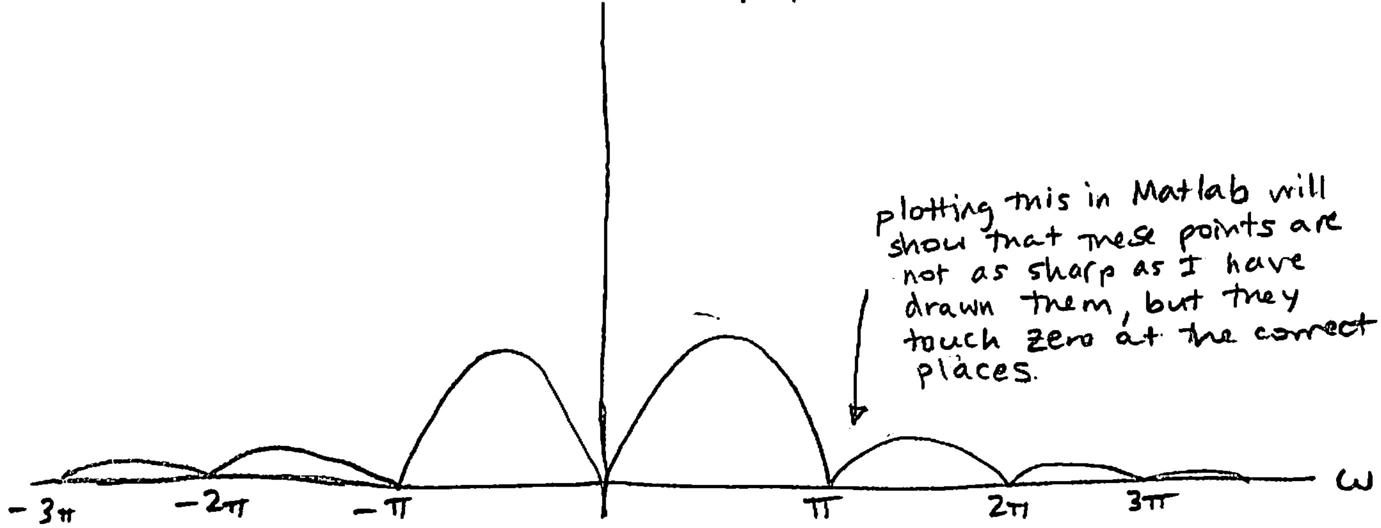
$$= \frac{1}{j\omega}\left[2 - \underbrace{\left(e^{j2\omega} + e^{-j2\omega}\right)}_{2\cos(2\omega)}\right]$$

$$= \frac{2}{j\omega}\left[1 - \cos(2\omega)\right] = \frac{4}{j\omega}\sin^2(\omega)$$

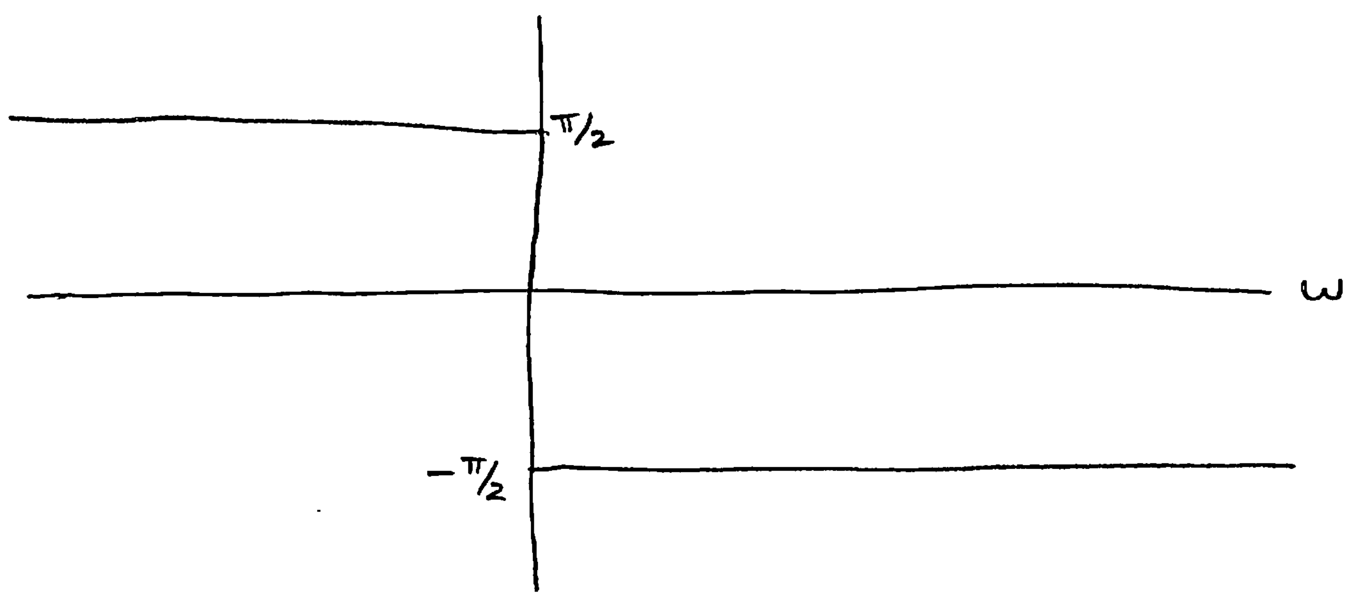
$$\Rightarrow \boxed{X_1(\omega) = \frac{-4j}{\omega}\sin^2(\omega)}$$

(b) 10 points. Carefully sketch the magnitude and phase spectra of the aperiodic signal $x_1(t)$.

$$|X_1(\omega)| = \frac{4}{|\omega|}\sin^2(\omega)$$



$$\measuredangle X_1(\omega) = \measuredangle \frac{-4j}{\omega} = \begin{cases} -\pi/2 & \omega > 0 \\ 0 & \omega = 0 \\ \pi/2 & \omega < 0 \end{cases}$$ (undefined really)



2. 30 points. Suppose you have the periodic signal $x_2(t)$ given in Figure 2 below.

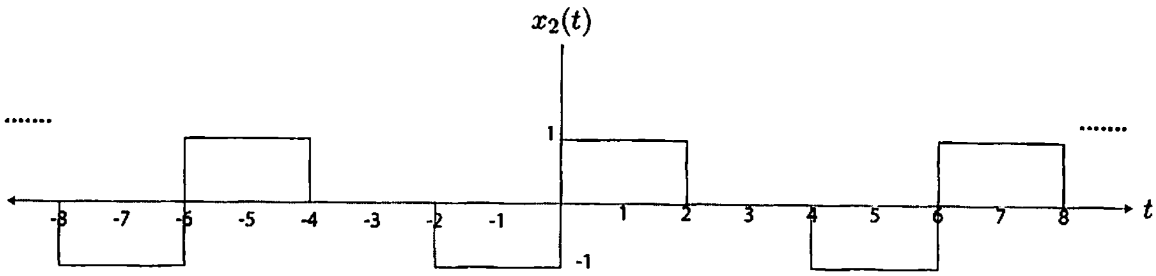


Figure 2: A periodic signal.

Using your result from Problem 1, write $x_1(t)$ as an *exponential* Fourier series and give an expression for the Fourier coefficients. **You do not need to perform any integrals to determine the Fourier coefficients.**

Here you just need to use the fact that

$$T_0 D_n = X_1(n\omega_0) \quad \text{where } T_0 = 6 \text{ and } \omega_0 = \frac{\pi}{3}$$

So $$D_n = \frac{-4j}{6n\frac{\pi}{3}} \sin^2\left(n\frac{\pi}{3}\right) = \frac{-2j}{n\pi} \sin^2\left(n\frac{\pi}{3}\right)$$

and

$$x_2(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\frac{\pi}{3}t}$$

$$\boxed{x_2(t) = \sum_{n=-\infty}^{\infty} \left(\frac{-2j}{n\pi} \sin^2\left(n\frac{\pi}{3}\right)\right) e^{jn\frac{\pi}{3}t}}$$

You can confirm these are the correct D_n also by doing the integration directly, i.e.

$$D_n = \frac{1}{T_0} \int_{T_0} x_2(t) e^{-jn\omega_0 t} dt$$

but this would be repeating the work you did in problem 1.

3. 30 points. Suppose you apply the periodic signal $x_2(t)$ as an input to a system with impulse response

$$h(t) = e^{-t}u(t) \tag{1}$$

where $u(t)$ is the usual unit step function. Using your results from Problem 2, write the output of this system $y(t)$ as an exponential Fourier series.

From the table on page 702 we have

$$e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}$$

hence $H(\omega) = \dfrac{1}{1+j\omega} = \dfrac{1-j\omega}{1+\omega^2}$

and $H(n\omega_0) = \dfrac{1-jn\frac{\pi}{3}}{1+(n\frac{\pi}{3})^2} = \dfrac{9-jn3\pi}{9+n^2\pi^2}$

The output of the system is then

$$y(t) = \sum_{n=-\infty}^{\infty} D_n H(n\omega_0) e^{jn\omega_0 t}$$

hence

$$y(t) = \sum_{n=-\infty}^{\infty} \frac{(-2j)(9-jn3\pi)}{(n\pi)(9+n^2\pi^2)} \sin^2\left(n\frac{\pi}{3}\right) e^{jn\frac{\pi}{3}t}$$

$$\boxed{y(t) = \sum_{n=-\infty}^{\infty} -\frac{(6\pi n + 18j)}{9\pi n + n^3\pi^3} \sin^2\left(n\pi/3\right) e^{jn\frac{\pi}{3}t}}$$