ECE 4304 HW 4

Problem 5.3

Consider the signals $s_1(t)$, $s_2(t)$, $s_3(t)$, and $s_4(t)$ shown in Fig. 1a. We wish to use the Gram-Schmidt orthogonalization procedure to find an orthonormal basis for this set of signals.

Step 1 We note that the energy of signal $s_1(t)$ is

$$E_1 = \int_0^T s_1^2(t)dt$$

$$= \int_0^{T/3} (1)^2 dt$$

$$= \frac{T}{3}$$

The first basis function $\phi_1(t)$ is therefore

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

$$= \left\{ \begin{array}{ll} \sqrt{3/T}, & 0 \le t \le T/3 \\ 0, & \text{otherwise} \end{array} \right\}$$

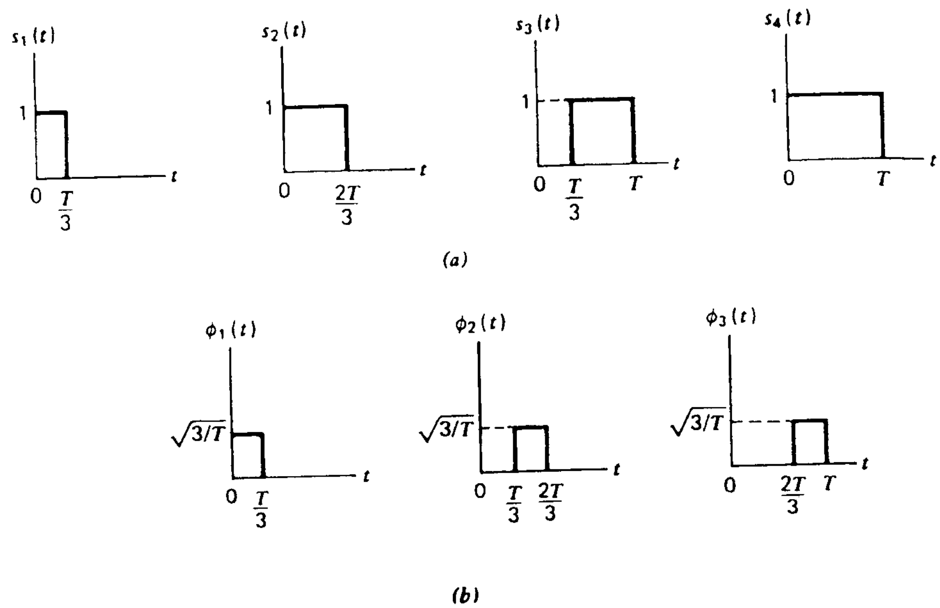


Figure 1

Step 2 Evaluating the projection of $s_2(t)$ onto $\phi_1(t)$, we find that

$$s_{21} = \int_0^T s_2(t)\phi_1(t)dt$$

$$= \int_0^{T/3} (1)\left(\sqrt{\frac{3}{T}}\right)dt$$

$$= \sqrt{\frac{3}{T}}$$

The energy of signal $s_2(t)$ is

$$E_2 = \int_0^T s_2^2(t)$$

$$= \int_0^{2T/3} (1)^2 dt$$

$$= \frac{2T}{3}$$

The second basis function $\phi_2(t)$ is therefore

$$\phi_2(t) = \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 - s_{21}^2}}$$

$$= \begin{cases} \sqrt{3/T}, & T/3 \le 2T/3 \\ 0 & \text{otherwise} \end{cases}$$

Step 3 Evaluating the projection of $s_3(t)$ onto $\phi_1(t)$,

$$s_{31} = \int_0^T s_3(t)\phi_1(t)dt$$

$$= 0$$

and the coefficient s_{32} equals

$$s_{32} = \int_0^T s_3(t)\phi_1(t)dt$$

$$= \int_{T/3}^{2T/3} (1)\left(\sqrt{\frac{3}{T}}\right)dt$$

$$= \sqrt{\frac{3}{T}}$$

The corresponding value of the intermediate function $g_i(t)$, with $i = 3$, is therefore

$$g_3(t) = s_3(t) - s_{31}\phi_1(t) - s_{32}\phi_2(t)$$

$$= \begin{cases} 1, & 2T/3 \le t \le T \\ 0, & \text{elsewhere} \end{cases}$$

Hence, the third basis function $\phi_3(t)$ is

$$\phi_3(t) = \frac{g_3(t)}{\sqrt{\int_0^T g_3^2(t)dt}}$$

$$= \begin{cases} \sqrt{3/T}, & 2T/3 \le t \le T \\ 0, & \text{elsewhere} \end{cases}$$

The orthogonalization process is now complete.

The three basis functions $\phi_1(t)$, $\phi_2(t)$, and $\phi_3(t)$ form an orthonormal set, as shown in Fig. 1b. In this example, we thus have $M = 4$ and $N = 3$, which means that the four signals $s_1(t)$, $s_2(t)$, $s_3(t)$, and $s_4(t)$ described in Fig. 1a do not form a linearly independent set. This is readily confirmed by noting that $s_4(t) = s_1(t) + s_3(t)$. Moreover, we note that any of these four signals can be expressed as a linear combination of the three basis functions, which is the essence of the Gram-Schmidt orthogonalization procedure.

Problem 5.5

Signals $s_1(t)$ and $s_2(t)$ are orthogonal to each other. The energy of $s_1(t)$ is

$$E_1 = \int_0^{T/2} 1^2 dt + \int_{T/2}^{T} (-1)^2 dt = T$$

The energy of $s_2(t)$ is

$$E_2 = \int_0^{T} 1^2 dt = T$$

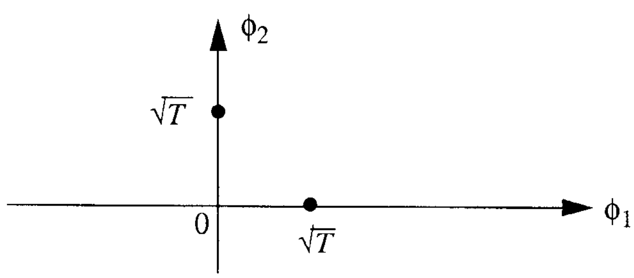
To represent the orthogonal signals $s_1(t)$ and $s_2(t)$, we need two basis functions. The first basis function is given by

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{s_1(t)}{\sqrt{T}}$$

The second basis function is given by

$$\phi_2(t) = \frac{s_2(t)}{\sqrt{E_2}} = \frac{s_2(t)}{\sqrt{T}}$$

The signal-space diagram for $s_1(t)$ and $s_2(t)$ is as shown below:



Problem 5.7

(a) The biorthogonal signals are defined as the negatives of orthogonal signals. Consider for example the two orthogonal signals $s_1(t)$ and $s_2(t)$ defined as follows:

$$s_1(t) = \sqrt{E}\phi_1(t)$$

$$s_2(t) = \sqrt{E}\phi_2(t)$$

where $\phi_1(t)$ and $\phi_2(t)$ are orthonormal basis functions. The biorthogonal signals are given by $-s_1(t)$ and $-s_2(t)$, which are respectively expressed in terms of the basis functions as $-\sqrt{E}\phi_1(t)$ and $-\sqrt{E}\phi_2(t)$. Hence, the inclusion of these two biorthogonal signals leaves the dimensionality of the signal-space diagram unchanged. This result holds for the general case of M orthogonal signals.

(b) The signal-space diagram for the biorthogonal signals corresponding to those shown in Fig. P5.5 is as shown in Fig. 1a. Incorporating this diagram with that of the solution to Problem 5.5, we get the 4-signal constellation shown in Fig. 1b.

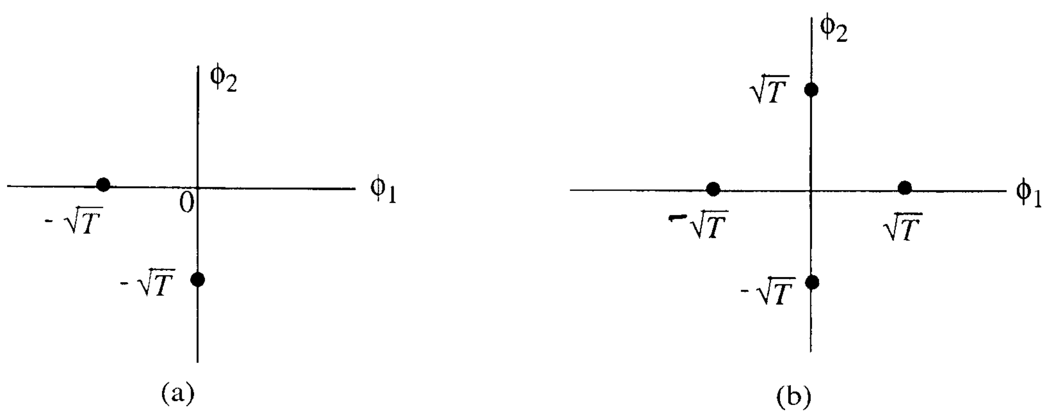


Figure 1

Problem 5.13

Energy of binary symbol 1 represented by signal $s_1(t)$ is

$$E_1 = \int_0^{T/2} (+1)^2 dt + \int_{T/2}^{T} (-1)^2 dt = T$$

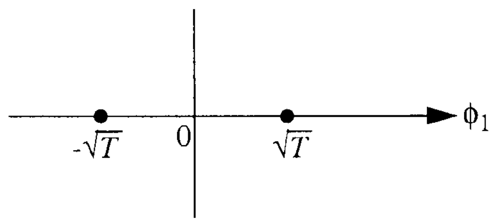
Energy of binary symbol 0 represented by signal $s_2(t)$ is the same as shown by

$$E_2 = \int_0^{T/2} (-1)^2 dt + \int_{T/2}^{T} (+1)^2 dt = T$$

The only basis function of the signal-space diagram is

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{s_1(t)}{\sqrt{T}}$$

The signal-space diagram of the Manchester code using the doublet pulse is as follows:

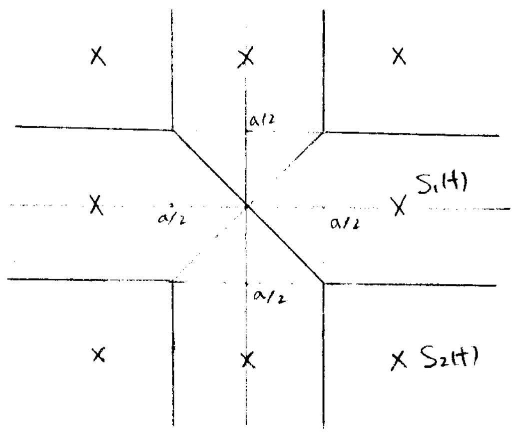


Hence, the distance between the two signal points is $d = 2\sqrt{T}$. The average probability of error over an AWGN channel is given by

$$P_e = \frac{1}{2}\text{erfc}\left(\frac{d}{2\sqrt{N_0}}\right) = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{T}{N_0}}\right) \qquad (5.89)$$

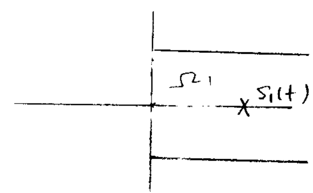
$$= Q\left(\sqrt{\frac{2T}{N_0}}\right)$$

4. a)



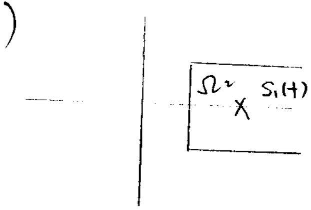
b) $\mathcal{E}_{b\text{-}av} = \dfrac{\mathcal{E}_{s_1} + \mathcal{E}_{s_2}}{2 \log_2 8} = \dfrac{1}{2} a^2$

c. $P(\text{correct} \mid S_1(t) \text{ transmitted}) := P_{s1}$



$P(V \text{ in } \Omega_2 \mid S_1(t)) < P_{s1} < P(V \text{ in } \Omega_1 \mid S_1(t))$

$P(V \text{ in } \Omega_1 \mid S_1(t)) = P(V_1 > 0, -\frac{a}{2} < V_2 < \frac{a}{2} \mid S_1(t))$



$= P(W_1 > -a) \; P(-\frac{a}{2} < W_2 < \frac{a}{2})$

$= \left[1 - Q\left(\sqrt{\frac{2a^2}{N_0}}\right)\right]\left[1 - 2Q\left(\sqrt{\frac{a^2}{2N_0}}\right)\right]$

$P(V \text{ in } \Omega_2 \mid S_1(t)) = P(V_1 > \frac{a}{2}, -\frac{a}{2} < V_2 < \frac{a}{2})$

$= P(W_1 > -\frac{a}{2}) P(-\frac{a}{2} < W_2 < \frac{a}{2})$

$= \left[1 - Q\left(\sqrt{\frac{a^2}{2N_0}}\right)\right]\left[1 - 2Q\left(\sqrt{\frac{a^2}{2N_0}}\right)\right]$

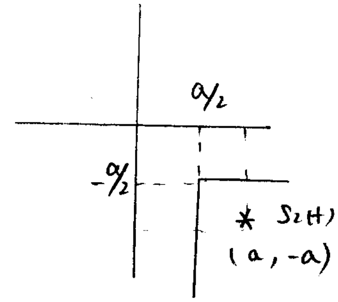
$= 1 - 3Q\left(\sqrt{\frac{a^2}{2N_0}}\right) + 2Q^2\left(\sqrt{\frac{a^2}{2N_0}}\right)$ $\quad a^2 = 2\mathcal{E}_{b\text{-}av}$

$1 - 3Q\left(\sqrt{\frac{\mathcal{E}_{b\text{-}av}}{N_0}}\right) + 2Q^2\left(\sqrt{\frac{\mathcal{E}_{b\text{-}av}}{N_0}}\right) < P_{s_1} < \left[1 - Q\left(\sqrt{\frac{4\mathcal{E}_{b\text{-}av}}{N_0}}\right)\right]\left[1 - 2Q\left(\sqrt{\frac{\mathcal{E}_{b\text{-}av}}{N_0}}\right)\right]$

$\uparrow$

$P(\text{correct} \mid S_1(t) \text{ transmitted})$

d) $P_{S2} := P(\text{correct} \mid S_2(t) \text{ transmitted})$



$$= P\left(V_1 > \frac{a}{2},\ V_2 < -\frac{a}{2} \,\middle|\, S_2(t)\right)$$
$$= P\left(W_1 > -\frac{a}{2}\right) P\left(W_2 < \frac{a}{2}\right)$$
$$= \left[1 - Q\left(\sqrt{\frac{a^2}{2N_0}}\right)\right]\left[1 - Q\left(\sqrt{\frac{a^2}{2N_0}}\right)\right]$$
$$= \left[1 - Q\left(\sqrt{\frac{\mathcal{E}_{b\text{-}av}}{N_0}}\right)\right]^2$$

e) $$P_{error} = \sum_{m=1}^{M} P(\underline{V} \in \bar{\Omega}_m,\ S_m(t) \text{ sent})$$

$\bar{\Omega}_m$ is the region in $\underline{V}$ space that cause an error if $S_m(t)$ sent

Hense $$P_{error} = \sum_{m=1}^{M} P(S_m(t) \text{ sent})\, P(\underline{V} \in \bar{\Omega}_m \mid S_m(t) \text{ sent})$$
$$= \sum_{m=1}^{M} \frac{1}{M}\left(1 - P(V \in \Omega_m \mid S_m(t) \text{ sent})\right)$$
$$= \sum_{m=1}^{M} \frac{1}{M}\left(1 - P(\text{correct} \mid S_m(t) \text{ sent})\right)$$

Due to the symetrical of the signal space, we notice that the 4 signals at corners has the same $P(\text{correct} \mid S_m(t) \text{ sent})$ and the 4 at the edges also has the same $P(\text{correct} \mid S_m(t) \text{ sent})$

Hense $$P_{error} = \frac{1}{8}\left(4(1 - P(\text{correct} \mid S_1(t) \text{ sent}) + 4(1 - P(\text{correct} \mid S_2 \text{ sent})\right)$$
$$= 1 - \frac{1}{2} P(\text{correct} \mid S_1 \text{ sent}) - \frac{1}{2} P(\text{correct} \mid S_2 \text{ sent})$$

lower bound on $P(\text{correct} \mid S_1(t) \text{ sent})$ is the upper bound on P_{error}

Hence $$P_{error} < 1 - \frac{1}{2}\left(1 - 3Q\left(\sqrt{\frac{\mathcal{E}_{b\text{-}av}}{N_0}}\right) + 2Q^2\left(\sqrt{\frac{\mathcal{E}_{b\text{-}av}}{N_0}}\right)\right)$$
$$- \frac{1}{2}\left[1 - 2Q\left(\sqrt{\frac{\mathcal{E}_{b\text{-}av}}{N_0}}\right) + Q^2\left(\sqrt{\frac{\mathcal{E}_{b\text{-}av}}{N_0}}\right)\right]$$

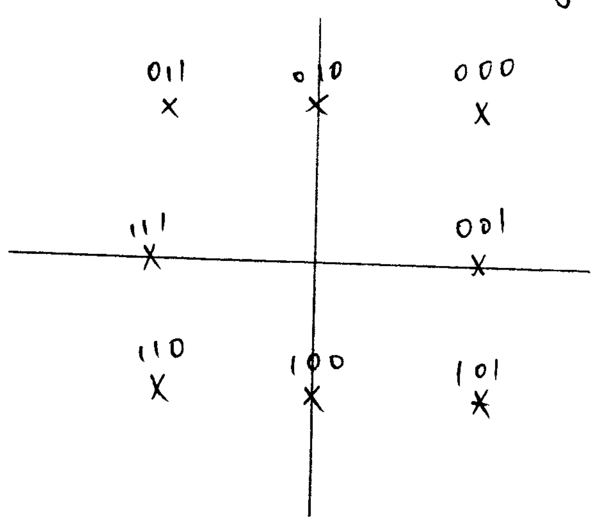
$$\boxed{P_{error} < \frac{5}{2} Q\left(\sqrt{\frac{\mathcal{E}_{b\text{-}av}}{N_0}}\right) - \frac{3}{2} Q^2\left(\sqrt{\frac{\mathcal{E}_{b\text{-}av}}{N_0}}\right)}$$

upper bound on $P(\text{correct} \mid s_1(t) \text{ sent})$ is the lower bound on P_{error}

Hense $$P_{error} > 1 - \frac{1}{2}\left[1 - Q\left(\sqrt{\frac{4\mathcal{E}_{b\_av}}{N_0}}\right)\right]\left[1 - 2Q\left(\sqrt{\frac{\mathcal{E}_{b\_av}}{N_0}}\right)\right] - \frac{1}{2}\left[1 - Q\left(\sqrt{\frac{\mathcal{E}_{b\_av}}{N_0}}\right)\right]^2.$$

$$\boxed{P_{error} > \frac{1}{2} Q\left(\sqrt{\frac{4\mathcal{E}_{b\_av}}{N_0}}\right) + 2Q\left(\sqrt{\frac{\mathcal{E}_{b\_av}}{N_0}}\right) - Q\left(\sqrt{\frac{4\mathcal{E}_{b\_av}}{N_0}}\right) Q\left(\sqrt{\frac{\mathcal{E}_{b\_av}}{N_0}}\right) - \frac{1}{2} Q^2\left(\sqrt{\frac{\mathcal{E}_{b\_av}}{N_0}}\right)}$$

(f) $$P_{error\_bit} \approx \frac{P_{error}}{\log_2 M} = \frac{P_{error}}{\log_2 8} = \frac{P_{error}}{3}$$



Problem 6.16

The probability of symbol error for 16-QAM is given by

$$P_e = 2\left(1-\frac{1}{\sqrt{M}}\right)\operatorname{erfc}\left(\sqrt{\frac{3E_{\text{av}}}{2(M-1)N_0}}\right) \qquad (6.64)$$

Setting $P_e = 10^{-3}$, we get

$$10^{-3} = 2\left(1-\frac{1}{4}\right)\operatorname{erfc}\left(\sqrt{\frac{3E_{\text{av}}}{30N_0}}\right)$$

Solving this equation for E_{av}/N_0,

$$\frac{E_{\text{av}}}{N_0} = 58 = 17.6\text{dB}$$

The probability of symbol error for 16-PSK is given by

$$P_e = \operatorname{erfc}\left(\sqrt{\frac{E}{N_0}}\sin(\pi/M)\right) \qquad (6.47)$$

Setting $P_e = 10^{-3}$, we get

$$10^{-3} = \operatorname{erfc}\left(\sqrt{\frac{E}{N_0}}\sin(\pi/16)\right)$$

Solving this equation for E/N_0, we get

$$\frac{E}{N_0} = 142 = 21.5\text{dB}$$

Hence, on the average, the 16-PSK demands 21.5 - 17.6 = 3.9 dB more symbol energy than the 16-QAM for $P_e = 10^{-3}$.

Thus the 16-QAM requires about 4 dB less in signal energy than the 16-PSK for a fixed N_0 and $P_e = 10^{-3}$, However, for this advantage of the 16-QAM over the 16-PSK to be realized, the channel must be linear.

Problem 6.21 If the Amplitude is 1 μV

The bit duration is

$$T_b = \frac{1}{2.5\times10^6\,\text{Hz}} = 0.4\ \mu s$$

The signal energy per bit is

$$E_b = \frac{1}{2} A_c^2 T_b$$
$$= \frac{1}{2}(10^{-6})^2 \times 0.4 \times 10^{-6} = 2 \times 10^{-19} \text{ joules}$$

(a) Coherent Binary FSK

The average probability of error is

$$P_e = \frac{1}{2}\,\text{erfc}(\sqrt{E_b/2N_0}) \qquad (6.102)$$

$$= \frac{1}{2}\,\text{erfc}(\sqrt{2\times10^{-19}/4\times10^{-20}})$$

$$= \frac{1}{2}\,\text{erfc}(\sqrt{5})$$

Using the approximation

$$\text{erfc}(u) \approx \frac{\exp(-u^2)}{\sqrt{\pi}\,u} \qquad (4.30)$$

we obtain the result

$$P_e = \frac{1}{2}\frac{\exp(-5)}{\sqrt{5\pi}} = 0.85 \times 10^{-3}$$

If the Amplitude is 1 mv

$$T_b = 0.4\ \mu s$$

$$E_b = \frac{1}{2} A_c^2 T_b$$
$$= 2\times10^{-13} \text{ joules}$$

$$P_e = \frac{1}{2}\,\text{erfc}(\sqrt{5\times10^6})$$

$$P_e = \frac{1}{2}\frac{\exp(-5\times10^6)}{\sqrt{5\times10^6\pi}} = 0$$