ECE 4304 HW5

Problem 9.2

Let the event $S=s_k$ denote the emission of symbol s_k by the source, Hence,

$$I(s_k) = \log_2\left(\frac{1}{p}\right) \text{ bits}$$

s_k	s_0	s_1	s_2	s_3
p_k	0.4	0.3	0.2	0.1
$I(s_k)$ bits	1.322	1.737	2.322	3.322

Entropy of the source is

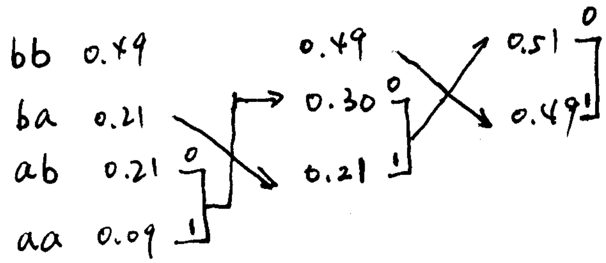
$$H(S) = p_0\log_2\left(\frac{1}{p_0}\right) + p_1\log_2\left(\frac{1}{p_1}\right) + p_2\log_2\left(\frac{1}{p_2}\right) + p_3\log_2\left(\frac{1}{p_3}\right)$$

$$= 0.4 \times 1.322 + 0.3 \times 1.737 + 0.2 \times 2.322 + 0.1 \times 3.322$$

$$= 1.8465 \text{ bits/symbol}$$

2. blocks of two

aa	ab	ba	bb
0.09	0.21	0.21	0.49



aa	001
ab	000
ba	01
bb	1

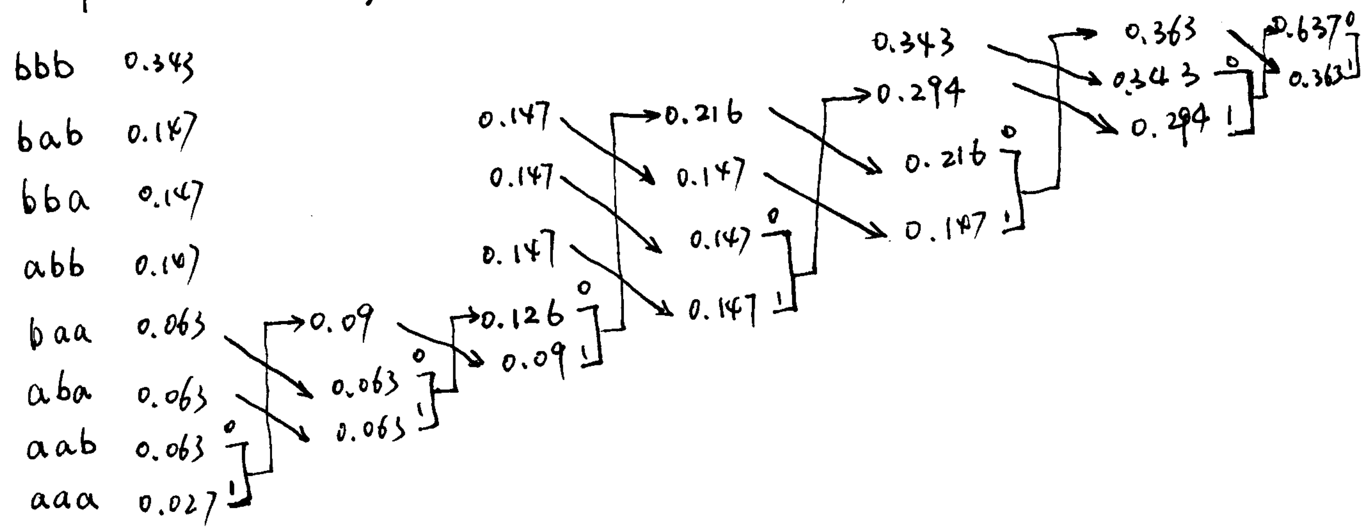
$\bar{L} = 0.49 \times 1 + 0.21 \times 2 + 0.21 \times 3 + 0.09 \times 3 = 1.81$

$H(\varphi^2) = 2H(\varphi) = 2(-0.3\log_2(0.3) - 0.7\log_2(0.7)) = 1.7626$

$\eta = \frac{H(\varphi^2)}{\bar{L}} = 97.38\%$

blocks of three

aaa	aab	aba	baa	abb	bba	bab	bbb
0.027	0.063	0.063	0.063	0.147	0.147	0.147	0.343



aaa	1011
aab	1010
aba	1001
baa	1000
abb	011
bba	010
bab	11
bbb	00

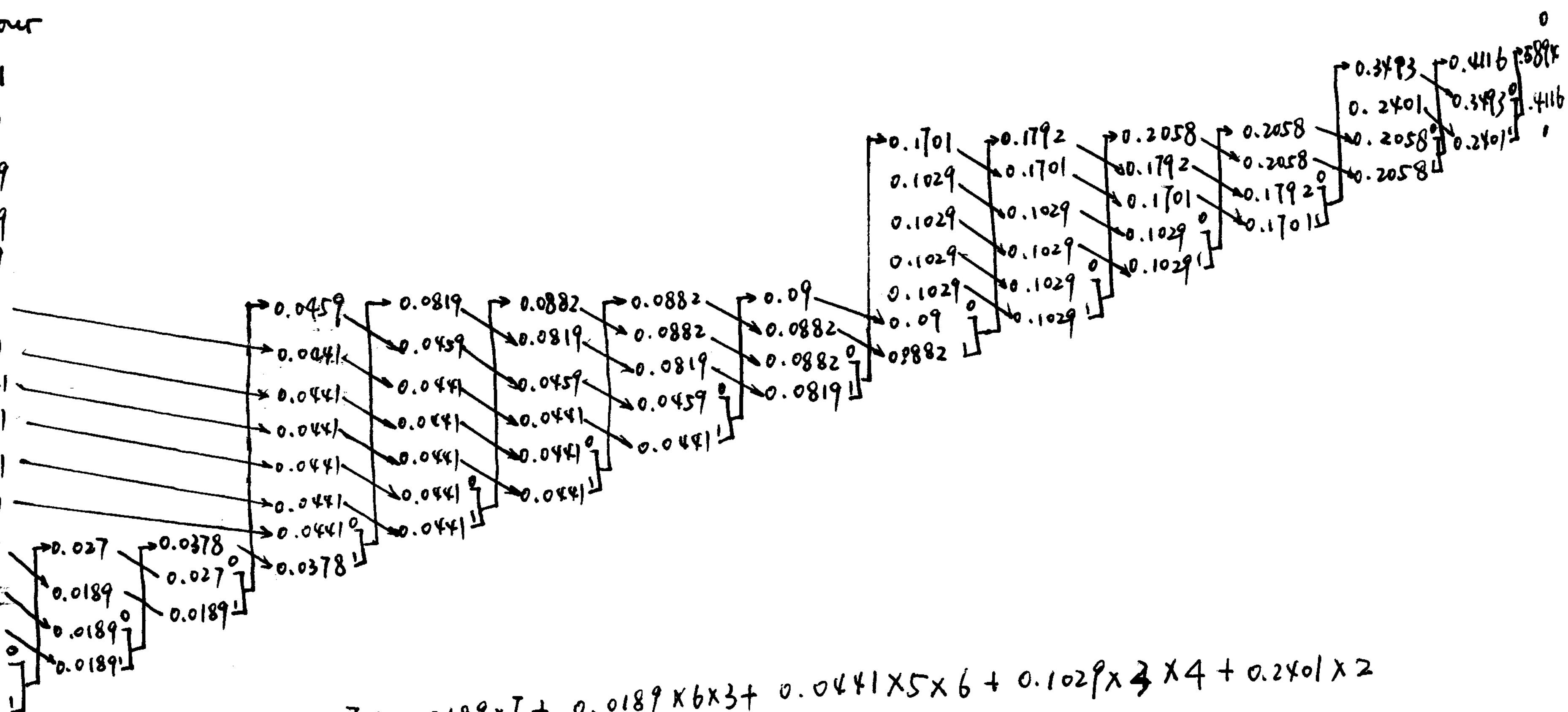
$\bar{L} = 0.027 \times 4 + 0.063 \times 4 \times 3 + 0.147 \times 3 \times 2 + 0.147 \times 2 + 0.343 \times 2$
$= 2.7260$

$H(\varphi^3) = 3H(\varphi) = 3(-0.3\log_2(0.3) - 0.7\log_2(0.7)) = 2.6439$

$\eta = \frac{H(\varphi^3)}{\bar{L}} = 96.99\%$

block of Four

bbbb	0.2401
bbba	0.1029
bbab	0.1029
babb	0.1029
abbb	0.1029
bbaa	0.0441
baba	0.0441
abba	0.0441
aabb	0.0441
abab	0.0441
baab	0.0441
aaab	0.0189
aaba	0.0189
abaa	0.0189
baaa	0.0189
aaaa	0.0081



bbbb	01
bbba	100
bbab	101
babb	110
abbb	111
bbaa	00001
baba	00010
abba	00011
aabb	00100
abab	00101
baab	00110
aaab	000001
aaba	001110
abaa	0[illegible]1111
baaa	01[illegible]000
aaaa	00[illegible]001

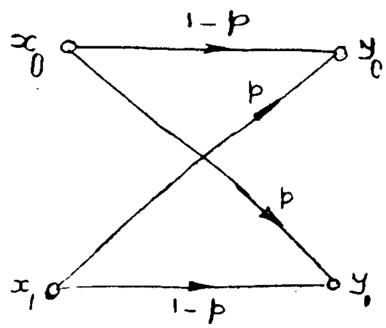
$$\bar{L} = 0.0081\times7 + 0.0189\times7 + 0.0189\times6\times3 + 0.0441\times5\times6 + 0.1029\times3\times4 + 0.2401\times2$$

$$= 3.5672$$

$$H(\varphi^4) = 4\,H(\varphi) = 4\times(-0.3\log_2(0.3) - 0.7\log_2(0.7)) = 3.5252$$

$$\eta = \frac{H(\varphi^x)}{\bar{L}} = 98.82\%$$

Problem 9.17



$$p(x_0) = p(x_1) = \frac{1}{2}$$

$$p(y_0) = (1 - p)p(x_0) + p\ p(x_1)$$

$$= (1 - p)\left(\frac{1}{2}\right) + p\left(\frac{1}{2}\right)$$

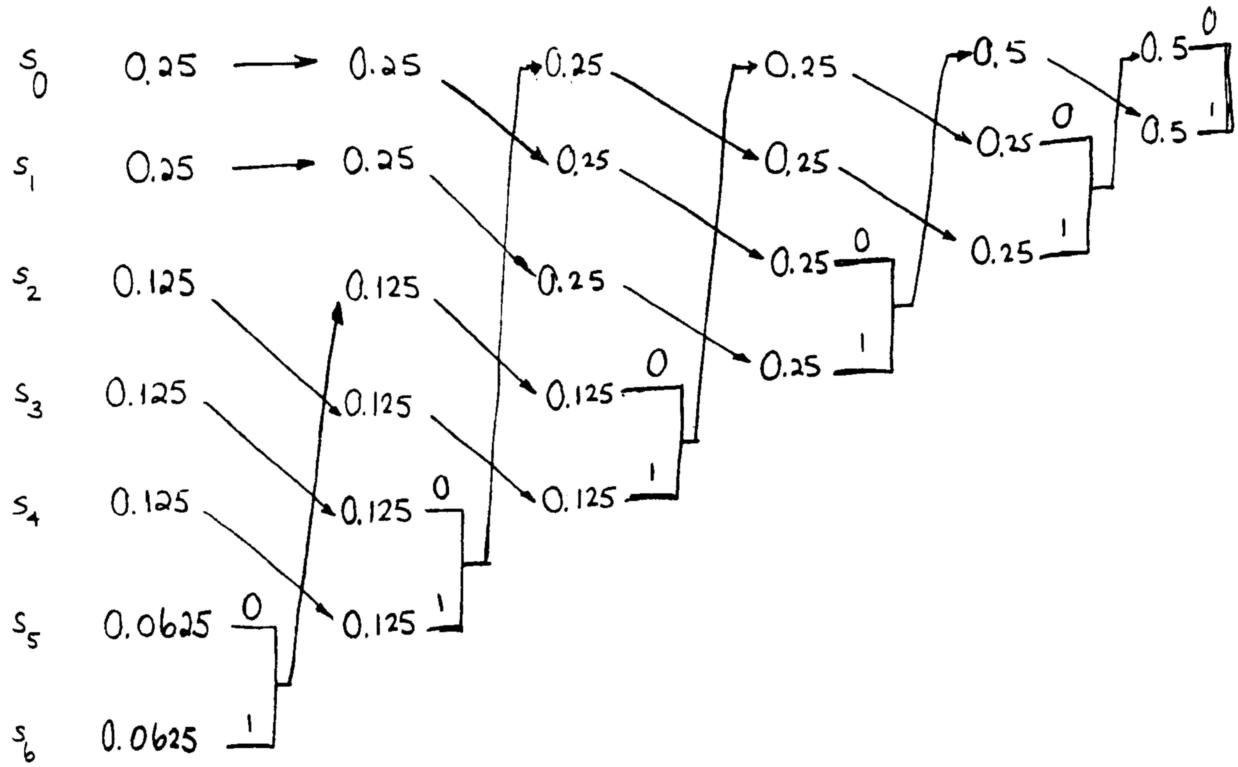
$$= \frac{1}{2}$$

$$p(y_1) = p\ p(x_0) + (1 - p)\ p(x_1)$$

$$= p\left(\frac{1}{2}\right) + (1 - p)\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}$$

Problem 9.12



The Huffman code is therefore

s_0	1 0
s_1	1 1
s_2	0 0 1
s_3	0 1 0
s_4	0 1 1
s_5	0 0 0 0
s_6	0 0 0 1

The average code-word length is

$$\bar{L} = \sum_{k=0}^{6} p_k l_k$$

$$= 0.25(2)(2) + 0.125(3)(3) + 0.0625(4)(2)$$

$$= 2.625$$

The entropy of the source is

$$H(S) = \sum_{k=0}^{6} p_k \log_2\left(\frac{1}{p_k}\right)$$

$$= 0.25(2) \log_2\left(\frac{1}{0.25}\right) + 0.125(3) \log_2\left(\frac{1}{0.125}\right)$$

$$+ 0.0625(2) \log_2\left(\frac{1}{0.0625}\right)$$

$$= 2.625$$

The efficiency of the code is therefore

$$\eta = \frac{H(S)}{\bar{L}} = \frac{2.625}{2.625} = 1$$

We could have shown that the efficiency of the code is 100% by inspection since

$$\eta = \frac{\sum_{k=0}^{6} p_k \log_2(1/p_k)}{\sum_{k=0}^{6} p_k l_k}$$

where $l_k = \log_2(1/p_k)$.

Problem 9.18

$$p(x_0) = \frac{1}{4}$$

$$p(x_1) = \frac{3}{4}$$

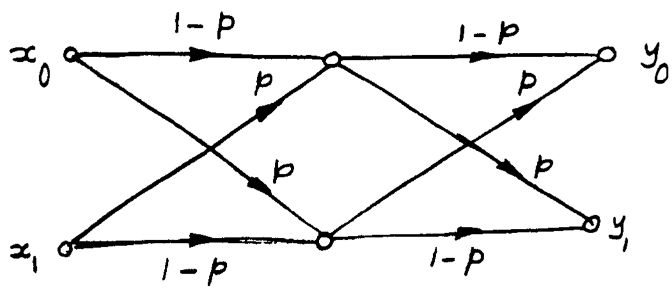
$$p(y_0) = (1 - p)\left(\frac{1}{4}\right) + p\left(\frac{3}{4}\right)$$

$$= \frac{1}{4} + \frac{p}{2}$$

$$p(y_i) = p\left(\frac{1}{4}\right) + (1 - p)\left(\frac{3}{4}\right)$$

$$= \frac{3}{4} - \frac{p}{2}$$

Problem 9.22

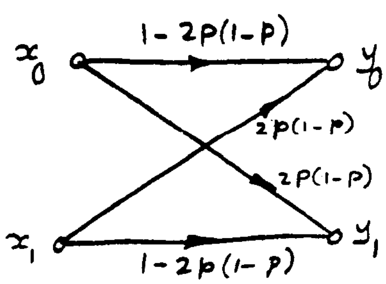


From this diagram, we obtain (by inspection)

$$P(y_0 \mid x_0) = (1 - p)^2 + p^2 = 1 - 2p(1 - p)$$

$$P(y_0 \mid x_1) = p(1 - p) + (1 - p)p = 2p(1 - p)$$

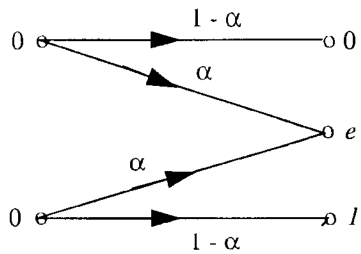
Hence, the cascade of two binary symmetric channels with a transition probability p is equivalent to a single binary symmetric channel with a transition probability equal to 2p(1 - p), as shown below:



Correspondingly, the channel capacity of the cascade is

$$\begin{aligned} C &= 1 - H(2p(1 - p)) \\ &= 1 - 2p(1 - p) \log_2 [2p(1 - p)] - (1 - 2p + 2p^2) \log_2 (1 - 2p + 2p^2) \end{aligned}$$

Problem 9.23



The mutual information for the erasure channel is

$$\mathbf{I(X;Y)} = \sum_{j=0}^{1}\sum_{k=0}^{2} p(x_j, y_k)\log_2\left(\frac{p(x_j, y_k)}{p(x_j)p(y_k)}\right) \tag{1}$$

The joint probabilities for the channel are

$$p(x_0, y_0) = (1-\alpha)p_0 \qquad p(x_1, y_0) = 0 \qquad p(x_0, y_2) = p_0\alpha$$

$$p(x_0, y_1) = 0 \qquad p(x_1, y_1) = (1-\alpha)p_1 \qquad p(x_1, y_2) = p_1\alpha$$

where $p_0 + p_1 = 1$. Substituting these values in (1), we get

$$\mathbf{I(X;Y)} = (1-\alpha)\left[p_0\log_2\left(\frac{1}{p_0}\right) + (1-p_0)\log_2\left(\frac{1}{1-p_0}\right)\right]$$

Since the transition probability $p = (1 - \alpha)$ is fixed, the mutual information $\mathbf{I(X;Y)}$ is maximized by choosing the a priori probability p_0 to maximize $H(p_0)$. This maximization occurs at $p_0 = 1/2$,. for which $H(p_0) = 1$. Hence, the channel capacity C of the erasure channel is $1 - \alpha$.