

# ECE4703: Lecture 9

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# FFT Complexity Analysis: $N = 1$

When  $N = 1$ , there is nothing to divide into even and odd parts, so we will just use the DFT. The DFT equation:

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \quad \text{for } k = 0, 1, \dots, N - 1$$

When  $N = 1$ , we have

$$X[0] = x[0]e^0 = x[0]$$

No multiplies or additions. Hence a one-point DFT/FFT has no MACs.

# FFT Complexity Analysis: $N = 2$

Two-point FFT computed using radix-2 decimation in time:

$$X[0] = X_{even}[0] + e^{-j2\pi \cdot 0/2} X_{odd}[0] = X_{even}[0] + X_{odd}[0]$$

$$X[1] = X_{even}[1] + e^{-j2\pi \cdot 1/2} X_{odd}[1] = X_{even}[1] - X_{odd}[1]$$

Note that  $X_{even}[k]$  and  $X_{odd}[k]$  for  $k = 0, 1$  are just one-point FFTs.

$$X_{even}[0] = \text{DFT}_1\{x[0], 0\} = x[0]$$

$$X_{odd}[0] = \text{DFT}_1\{x[1], 0\} = x[1]$$

$$X_{even}[1] = \text{DFT}_1\{x[0], 1\} = x[0]$$

$$X_{odd}[1] = \text{DFT}_1\{x[1], 1\} = x[1]$$

Hence,

$$X[0] = x[0] + x[1]$$

$$X[1] = x[0] - x[1]$$

How many MACs in the R2-DIT two-point FFT?

Note that the two-point DFT and FFT are identical. Still no gain.

# An Observation

Look again at the one-point FFTs:

$$X_{\text{even}}[0] = \text{DFT}_1\{x[0], 0\} = x[0]$$

$$X_{\text{odd}}[0] = \text{DFT}_1\{x[1], 0\} = x[1]$$

$$X_{\text{even}}[1] = \text{DFT}_1\{x[0], 1\} = x[0]$$

$$X_{\text{odd}}[1] = \text{DFT}_1\{x[1], 1\} = x[1]$$

Note that  $X_{\text{even}}[1] = X_{\text{even}}[0]$  and  $X_{\text{odd}}[1] = X_{\text{odd}}[0]$ .

This is a consequence of the **periodicity** of the DFT/FFT. Recall that, in general, a  $P$ -point DFT/FFT satisfies

$$X[k + \ell P] = \text{DFT}_P\{\{x\}, k + \ell P\} = X[k]$$

This property is **critical** to getting the desired complexity reduction.

# FFT Complexity Analysis: $N = 4$ (part 1)

Four-point FFT computed using radix-2 decimation in time:

$$X[0] = X_{\text{even}}[0] + e^{-j2\pi \cdot 0/4} X_{\text{odd}}[0] = X_{\text{even}}[0] + X_{\text{odd}}[0]$$

$$X[1] = X_{\text{even}}[1] + e^{-j2\pi \cdot 1/4} X_{\text{odd}}[1] = X_{\text{even}}[1] - jX_{\text{odd}}[1]$$

$$X[2] = X_{\text{even}}[2] + e^{-j2\pi \cdot 2/4} X_{\text{odd}}[2] = X_{\text{even}}[2] - X_{\text{odd}}[2]$$

$$X[3] = X_{\text{even}}[3] + e^{-j2\pi \cdot 3/4} X_{\text{odd}}[3] = X_{\text{even}}[3] + jX_{\text{odd}}[3]$$

Remarks:

- ▶  $X_{\text{even}}[k]$  and  $X_{\text{odd}}[k]$  for  $k = 0, 1, 2, 3$  are all results from two-point FFTs.
- ▶ Recall that the two-point DFT/FFT  $\{X[0], X[1]\} = \text{FFT}_2\{x[0], x[1]\}$  requires two MACs to return **two values**.
- ▶ The **periodicity** of the DFT/FFT implies that

$$X_{\text{even}}[2] = X_{\text{even}}[0]$$

$$X_{\text{even}}[3] = X_{\text{even}}[1]$$

$$X_{\text{odd}}[2] = X_{\text{odd}}[0]$$

$$X_{\text{odd}}[3] = X_{\text{odd}}[1]$$

Hence, **we only need to compute**:  $\{X_{\text{even}}[0], X_{\text{even}}[1]\}, \{X_{\text{odd}}[0], X_{\text{odd}}[1]\}$ .

# FFT Complexity Analysis: $N = 4$ (part 2)

Four-point FFT computed using radix-2 decimation in time:

$$X[0] = X_{even}[0] + e^{-j2\pi \cdot 0/4} X_{odd}[0] = X_{even}[0] + X_{odd}[0]$$

$$X[1] = X_{even}[1] + e^{-j2\pi \cdot 1/4} X_{odd}[1] = X_{even}[1] - jX_{odd}[1]$$

$$X[2] = X_{even}[2] + e^{-j2\pi \cdot 2/4} X_{odd}[2] = X_{even}[2] - X_{odd}[2]$$

$$X[3] = X_{even}[3] + e^{-j2\pi \cdot 3/4} X_{odd}[3] = X_{even}[3] + jX_{odd}[3]$$

Computation of  $\{X_{even}[0], X_{even}[1]\} = \text{FFT}_2\{x[0], x[2]\}$  requires how many MACs? 2

Computation of  $\{X_{odd}[0], X_{odd}[1]\} = \text{FFT}_2\{x[1], x[3]\}$  requires how many MACs? 2

How many more MACs are required to assemble the results into a four-point FFT? 4

Hence, **the total MACs needed to compute a four-point FFT is 8.**

# FFT Complexity Analysis: $N = 8$

Eight-point FFT computed using radix-2 decimation in time follows the same accounting:

$$X[k] = X_{even}[k] + e^{-j2\pi \cdot k/8} X_{odd}[k] \quad \text{for } k = 0, 1, \dots, 7$$

Computation of

$\{X_{even}[0], X_{even}[1], X_{even}[2], X_{even}[3]\} = \text{FFT}_4\{x[0], x[2], x[4], x[6]\}$   
requires how many MACs? 8

Computation of

$\{X_{odd}[0], X_{odd}[1], X_{odd}[2], X_{odd}[3]\} = \text{FFT}_4\{x[1], x[3], x[5], x[7]\}$  requires  
how many MACs? 8

How many more MACs are required to assemble the results into an eight-point FFT? 8

Hence, **the total MACs needed to compute an eight-point FFT is 24.**

# FFT Complexity Analysis: General $N$

OPERATION	MACs
Computation of $N$ one-point FFTs	0
Assembling $\frac{N}{2}$ two-point FFTs from $N$ one-point FFTs	$N$
Assembling $\frac{N}{4}$ four-point FFTs from $\frac{N}{2}$ two-point FFTs	$N$
Assembling $\frac{N}{8}$ eight-point FFTs from $\frac{N}{4}$ four-point FFTs	$N$
$\vdots$	$\vdots$
Assembling one $N$ -point FFT from two $\frac{N}{2}$ -point FFTs	$N$

Hence, the total MACs needed to compute an  $N$ -point FFT is \_\_\_\_\_.