

ECE503 Homework Assignment Number 2

Due by 8:50pm on Monday 30-Jan-2012

IMPORTANT: Please place your ECE mailbox number on all homework assignments. Your ECE mailbox number can be found on the course web page.

Make sure your reasoning and work are clear to receive full credit for each problem. Points will be deducted for a disorderly presentation of your solution. Please also refer to the course academic honesty policies regarding collaboration on homework assignments.

1. 4 points. Mitra 4.3.
2. 2 points. Mitra 4.22.
3. 3 points. Mitra 4.40.
4. 3 points. Mitra 4.42. You can use Matlab to verify your answer.
5. 3 points. Mitra 4.45. You can use Matlab to verify your answer.
6. 3 points. Mitra 4.64. You can use Matlab to verify your answer for particular values of α and β .
7. 3 points. Mitra 4.69. Use Matlab to verify your filter blocks the low frequency component.
8. 4 points. Mitra M4.3.

1. Mitra 4.3

(a) Given $y[n] = x[n+3]$. For an input $x_i[n]$, $i = 1, 2$, the output is $y_i[n] = x_i[n+3]$, $i = 1, 2$.

Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is:

$$y[n] = Ax_1[n+3] + Bx_2[n+3] = Ay_1[n] + By_2[n].$$

Hence the system is linear. For an input $x[n] = \delta[n]$, the output is the impulse response $h[n] = \delta[n+3]$. For $n = 3$, $h[n] = \delta[0] = 1$. Thus, $h[n] \neq 0$ for $n < 0$. Hence, the system is not causal.

For a bounded input $|x[n]| \leq B < \infty$, the magnitude of the output samples are:
 $|y[n]| = B < \infty$.

As the output is also a bounded sequence, the system is BIBO stable. Finally, let $y[n]$ and $y_1[n]$ be the outputs for inputs $x[n]$ and $x_1[n]$, respectively. If:

$$x_1[n] = x[n - n_o] \quad \text{then} \quad y_1[n] = x[n - n_o + 3] = y[n - n_o].$$

Hence, the system is time-invariant.

(b) Given $y[n] = x[2-n] + \alpha$ with α nonzero constant.

For an input $x_i[n]$, $i = 1, 2$, the output is $y_i[n] = x_i[2-n] + \alpha$, $i = 1, 2$. Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is:

$$y[n] = x[2-n] + \alpha = Ax_1[2-n] + Bx_2[2-n] + \alpha.$$

On the other hand

$$Ay_1[n] + By_2[n] = Ax_1[2-n] + A\alpha + Bx_2[2-n] + B\alpha \neq y[n].$$

Hence the system is nonlinear.

For an input $x[n] = \delta[n]$, the output is the impulse response $h[n] = \delta[2-n] + \alpha$. For $n < 2$, $h[n] = \alpha$. Thus, the system is non-causal.

For a bounded input $|x[n]| \leq B < \infty$, the magnitude of the output samples are:

$$|y[n]| = |B + \alpha| < \infty.$$

As the output is also a bounded sequence, the system is BIBO stable.

Finally, let $y[n]$ and $y_1[n]$ be the outputs for inputs $x[n]$ and $x_1[n]$, respectively. Let $x[n] = n$. Then, $y[n] = 2 - n + \alpha$. Let $x_1[n] = x[n-1] = n-1$. Then, $y_1[n] = x_1[2-n] + \alpha = 3 - n + \alpha$. On the other hand, $y[n-1] = 2 - n - 1 + \alpha = 1 - n + \alpha \neq y_1[n]$.

Hence, the system is not time-invariant.

(c) Given $y[n] = \ln(1 - |x[n]|)$. For an input $x_i[n]$, $i = 1, 2$, the output is $y_i[n] = \ln(1 - |x_i[n]|)$.

For an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is:

$$y[n] = \ln(1 - |Ax_1[n] + Bx_2[n]|) \neq Ay_1[n] + By_2[n].$$

Hence the system is nonlinear.

For an input $x[n] = \delta[n]$, the output is the impulse response $h[n] = \ln(1 - |\delta[n]|)$. For $n < 0$, $h[n] = \ln(1) = 0$. Thus, the system is causal.

For a bounded input $|x[n]| \leq B < \infty$, the magnitude of the output samples are $|y[n]| \leq \ln(1 - B) < \infty$. As the output is also a bounded sequence, the system is BIBO stable.

(c) Given $y[n] = \ln(1 - |x[n]|)$. For an input $x_i[n]$, $i = 1, 2$, the output is $y_i[n] = \ln(1 - |x_i[n]|)$.

For an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is:

$$y[n] = \ln(1 - |Ax_1[n] + Bx_2[n]|) \neq Ay_1[n] + By_2[n].$$

Hence the system is nonlinear.

For an input $x[n] = \delta[n]$, the output is the impulse response $h[n] = \ln(1 - |\delta[n]|)$. For $n < 0$, $h[n] = \ln(1) = 0$. Thus, the system is causal.

For a bounded input $|x[n]| \leq B < \infty$, the magnitude of the output samples are $|y[n]| \leq \ln(1 - B) < \infty$. As the output is also a bounded sequence, the system is BIBO stable.

Finally, let $y[n]$ and $y_1[n]$ be the outputs for inputs $x[n]$ and $x_1[n]$, respectively. If $x_1[n] = x[n - n_o]$ then: $y_1[n] = \ln(1 - |x[n - n_o]|) = y[n - n_o]$.

Hence, the system is time-invariant.

(d) Given $y[n] = \beta + \sum_{\ell=-1}^3 x[n - \ell]$, with β a nonzero constant. For an input $x_i[n]$, $i = 1, 2$, the output is: $y_i[n] = \beta + \sum_{\ell=-1}^3 x_i[n - \ell]$.

Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is:

$$y[n] = \beta + \sum_{\ell=-1}^3 (Ax_1[n - \ell] + Bx_2[n - \ell]) = \beta + \sum_{\ell=-1}^3 Ax_1[n - \ell] + \sum_{\ell=-1}^3 Bx_2[n - \ell] \neq Ay_1[n] + By_2[n].$$

Hence the system is nonlinear.

$y[0] = \beta + \sum_{\ell=-1}^3 x[0 - \ell] = \beta + x[3] + x[-2] + x[0] + x[1]$. Since the output at $n = 0$ depends on the future value of $x[n]$ at $n = 1$, the system is causal.

For a bounded input $|x[n]| \leq B < \infty$, the magnitude of the output samples are $|y[n]| \leq \beta + 5B < \infty$. As the output is also a bounded sequence, the system is BIBO stable.

Finally, let $y[n]$ and $y_1[n]$ be the outputs for inputs $x[n]$ and $x_1[n]$, respectively. If

$$x_1[n] = x[n - n_o] \text{ then: } y_1[n] = \beta + \sum_{\ell=-1}^3 x[n - n_o - \ell] = y[n - n_o].$$

Hence, the system is time-invariant.

2. Mitra 4.22

The convolution with the periodic sequence is given by: $y[n] = \sum_{m=-\infty}^{\infty} h[m]\tilde{x}[n-m]$.

$$\text{Hence: } y[n+kN] = \sum_{m=-\infty}^{\infty} h[m]\tilde{x}[n+kN-m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] = y[n].$$

Thus, $y[n]$ is also a periodic sequence with a period N .

3. Mitra 4.40

The difference equation for a first order complex coefficient digital filter is given by:

$$y[n] = \alpha y[n-1] + x[n].$$

Denoting $y[n] = y_{re}[n] + j y_{im}[n]$, and $\alpha = a + jb$, we get:

$$y_{re}[n] + j y_{im}[n] = (a + jb)(y_{re}[n-1] + j y_{im}[n-1]) + x[n].$$

Equating the real and the imaginary parts, and noting that $x[n]$ is real, we get:

$$\begin{aligned} y_{re}[n] &= a y_{re}[n-1] - b y_{im}[n-1] + x[n], \\ y_{im}[n] &= b y_{re}[n-1] + a y_{im}[n-1]. \end{aligned}$$

From the second equation we have: $y_{im}[n-1] = \frac{1}{a} y_{im}[n] - \frac{b}{a} y_{re}[n-1]$.

Substituting this equation in the top left equation we arrive at

$$y_{re}[n] = a y_{re}[n-1] - \frac{b}{a} y_{im}[n] + \frac{b^2}{a} y_{re}[n-1] + x[n],$$

From this we get: $b y_{im}[n-1] = -a y_{re}[n-1] + (a^2 + b^2) y_{re}[n-2] + a x[n-1]$.

Substituting the above equation in the equation $y_{re}[n] = a y_{re}[n-1] - b y_{im}[n-1] + x[n]$

we arrive at: $y_{re}[n] = 2a y_{re}[n-1] - (a^2 + b^2) y_{re}[n-2] + x[n] - a x[n-1]$.

This is a second-order difference equation representing $y_{re}[n]$ in terms of $x[n]$.

4. Mitra 4.42

The difference equation to solve is given by: $y[n] - 0.16y[n-1] = 5.88\mu[n]$, $y[-1] = 5$.

The total solution is given by: $y[n] = y_c[n] + y_p[n]$, where, $y_c[n]$ is the complementary solution and $y_p[n]$ is the particular solution. The complementary solution is obtained by solving: $y_c[n] - 0.16y_c[n-1] = 0$.

To this end we set $y_c[n] = \alpha\lambda^n$, which yields: $\alpha\lambda^n - 0.16\alpha\lambda^{n-1} = 0$, or $\alpha\lambda^n \left(1 - \frac{0.16}{\lambda}\right) = 0$.

This implies that $\lambda = 0.16$, hence $y_c[n] = \alpha(0.16)^n$.

For the particular solution we choose $y_p[n] = \beta$. Substituting this solution in the difference equation representing the system we get: $\beta - 0.16\beta = 5.88\mu[n]$.

Setting $n = 0$ we get $\beta - 0.16\beta = 5.88$ and hence $\beta = 5.88/0.84 = 7$. Therefore:

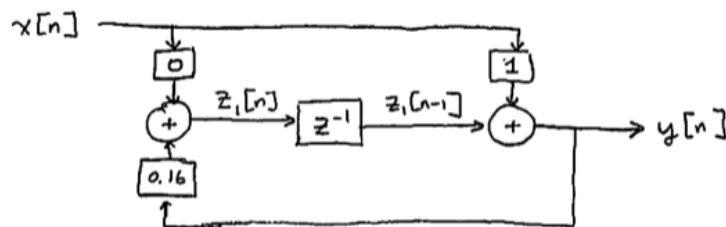
$$y[n] = y_c[n] + y_p[n] = \alpha(0.16)^n + 7.$$

For $n = -1$, we thus have $y[-1] = 5 = \alpha(0.16)^{-1} + 7$.

This implies $\alpha = -0.32$ and thus the total solution is thus given by:

$$y[n] = -0.32(0.16)^n + 7, \quad n \geq 0.$$

To verify this is correct in Matlab, we have to first convert the initial conditions as specified in the problem to the format that Matlab wants. The filter structure is pretty simple this time:



We need $z_1[-1]$ but we have $y[-1] = 5$

Note $y[0] = z_1[-1] + x[0]$ (from the block diagram)

From the DE, we know $y[0] = 0.16y[-1] + 5.88$

$$\begin{aligned} \text{so } z_1[-1] &= y[0] - x[0] = 0.16 * 5 + 5.88 - 5.88 \\ &= 0.8 \end{aligned}$$

Hence $z1[-1] = 0.8$, which we use in the Matlab script below.

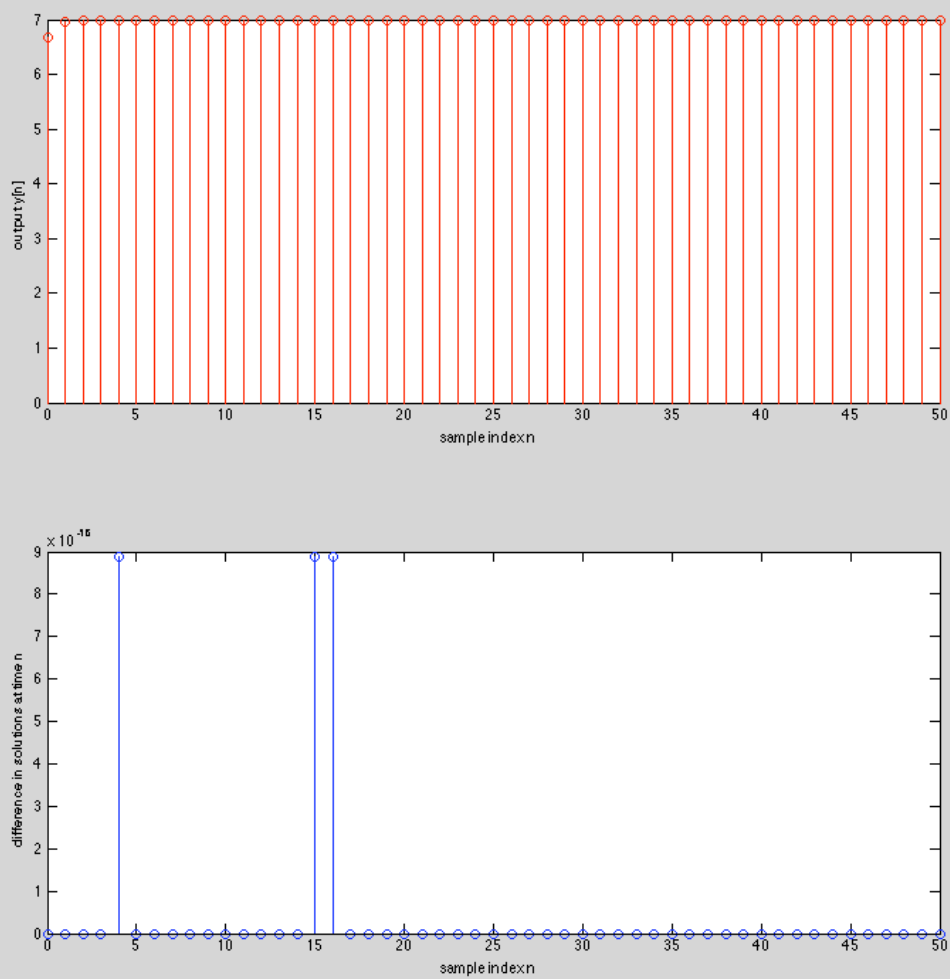
```
% Matlab verification of Mitra 4.42
% DRB 24-Jan-2012
N = 50;                % maximum sample index
n = 0:N;               % sample indices

b = 1;                 % feedforward coefficients
a = [1 -0.16];         % feedback coefficients
x = 5.88*ones(1,N+1);  % input signal
zi = 0.8;
ym = filter(b,a,x,zi);

y = -0.32*(0.16).^n+7; % analytical solution

subplot(2,1,1);
stem(n,y);
hold on
stem(n,ym,'r');
hold off
xlabel('sample index n');
ylabel('output y[n]');
subplot(2,1,2);
stem(n,y-ym);
xlabel('sample index n');
ylabel('difference in solutions at time n');
```

The results are shown on the next page.



Note the “spikes” in the error plot on the bottom are on the order of $1\text{E-}15$, which we can attribute to computational precision effects.

5. Mitra 4.45

We can use the complementary solution from Problem 4.42 in order to find the impulse response, as described in Section 4.6.3: $h[n] = \alpha(0.16)^n$.

To solve for α , we set $n = 0$ while letting $x[n] = \delta[n]$ and get: $h[0] = \alpha(0.16)^0 = \alpha = 1$.

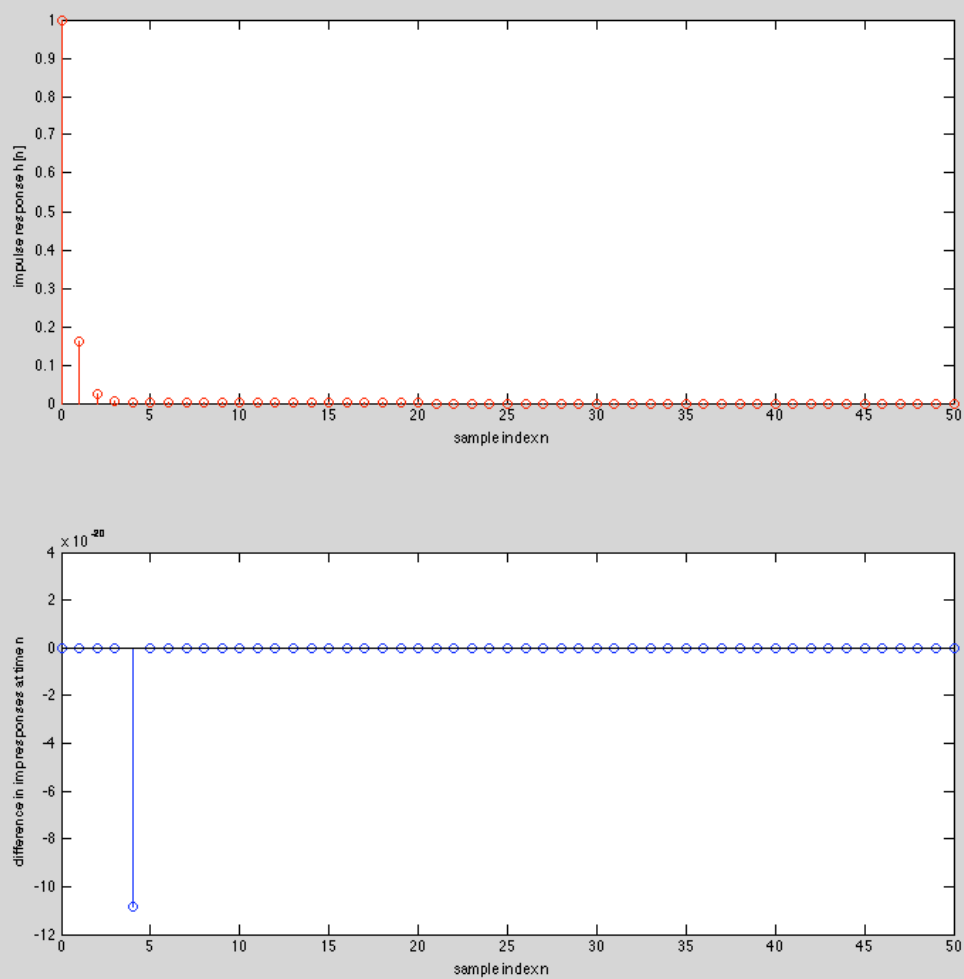
Therefore, the impulse response is: $h[n] = (0.16)^n$.

To check this in Matlab, we can extend our code for the previous problem by adding the following lines:

```
% impulse response
x = [1 zeros(1,N)];      % impulse
zi = 0;                  % relaxed conditions
hm = filter(b,a,x,zi);
h = 0.16.^n;

subplot(2,1,1);
stem(n,h);
hold on
stem(n,hm,'r');
hold off
xlabel('sample index n');
ylabel('impulse response h[n]');
subplot(2,1,2);
stem(n,h-hm);
xlabel('sample index n');
ylabel('difference in imp responses at time n');
```

The results are shown on the next page.



Again, the error “spikes” are very small (around $1\text{E-}19$), which we can attribute to computational precision effects.

6. Mitra 4.64

Given two LTI systems: $h_1[n] = \alpha\delta[n] + \delta[n-1]$ and $h_2[n] = \beta^n \mu[n]$.

The impulse response of the cascade is given by

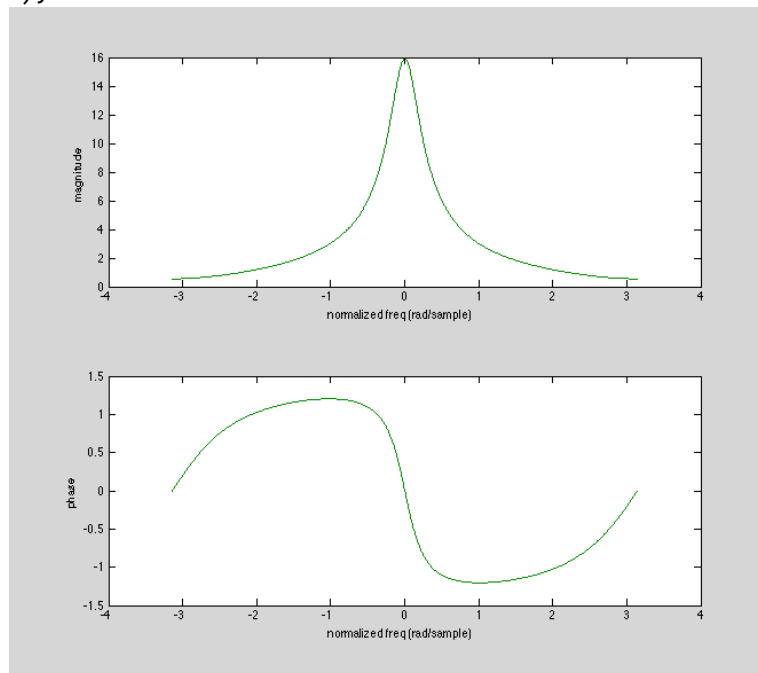
$$h[n] = h_1[n] \otimes h_2[n] = (\alpha\delta[n] + \delta[n-1]) \otimes (\beta^n \mu[n]).$$

The frequency response of the cascade is thus

$$H(e^{j\omega}) = (\alpha + e^{-j\omega}) \left(\frac{1}{1 - \beta e^{-j\omega}} \right) = \left(\frac{\alpha + e^{-j\omega}}{1 - \beta e^{-j\omega}} \right).$$

We can verify our analysis above for particular values of alpha and beta as follows:

```
% Matlab verification of Mitra 4.64
% DRB 24-Jan-2012
alpha = 2;
beta = 2*rand-1;
b = [alpha 1];
a = [1 -beta];
w = linspace(-pi,pi,1024);
H = (alpha+exp(-1i*w))./(1-beta*exp(-1i*w)); % analytical solution
Hm = freqz(b,a,w); % matlab solution
subplot(2,1,1)
plot(w,abs(H),w,abs(Hm));
xlabel('normalized freq (rad/sample)');
ylabel('magnitude');
subplot(2,1,2)
plot(w,unwrap(angle(H)),w,unwrap(angle(Hm)));
xlabel('normalized freq (rad/sample)');
ylabel('phase');
```



We see the analytical and Matlab solutions agree perfectly.

The following shows the analysis of the relationship between alpha and beta to cause $|H(\omega)| = 1$.

$$H(\omega) = \frac{\alpha + e^{-j\omega}}{1 - \beta e^{-j\omega}}$$

$$\begin{aligned} \text{We want } |H(\omega)| = 1 &\Leftrightarrow |H(\omega)|^2 = 1 \\ &\Leftrightarrow \frac{(\alpha_R + \cos(\omega))^2 + (\alpha_I - \sin(\omega))^2}{(1 + \gamma_R)^2 + (0 + \gamma_I^2)} = 1 \end{aligned}$$

$$\text{where } \gamma_R = -(\beta_R \cos(\omega) + \beta_I \sin(\omega))$$

$$\gamma_I = -(\beta_R (-\sin(\omega)) + \beta_I \cos(\omega))$$

simplifying...

$$\begin{aligned} \text{Numerator: } &\alpha_R^2 + 2\alpha_R \cos(\omega) + \cos^2(\omega) + \alpha_I^2 - 2\alpha_I \sin(\omega) + \sin^2(\omega) \\ &= \alpha_R^2 + \alpha_I^2 + 2\alpha_R \cos(\omega) - 2\alpha_I \sin(\omega) + 1 \\ &= |\alpha|^2 + 1 + 2\alpha_R \cos(\omega) - 2\alpha_I \sin(\omega) \end{aligned}$$

$$\begin{aligned} \text{Denominator: } &(1 - \beta_R \cos(\omega) - \beta_I \sin(\omega))^2 + (\beta_I \cos(\omega) - \beta_R \sin(\omega))^2 \\ &= (1 + \beta_R^2 \cos^2(\omega) + \beta_I^2 \sin^2(\omega) - 2\beta_R \cos(\omega) - 2\beta_I \sin(\omega) \\ &\quad + 2\beta_R \beta_I \cos(\omega) \sin(\omega) + \beta_I^2 \cos^2(\omega) + \beta_R^2 \sin^2(\omega) \\ &\quad - 2\beta_I \beta_R \cos(\omega) \sin(\omega)) \\ &= 1 + \beta_R^2 + \beta_I^2 - 2\beta_R \cos(\omega) - 2\beta_I \sin(\omega) \\ &= |\beta|^2 + 1 - 2\beta_R \cos(\omega) - 2\beta_I \sin(\omega) \end{aligned}$$

Note that numerator = denominator if $\alpha_R = -\beta_R$ and $\alpha_I = \beta_I$
 in other words: if $\beta = a + jb$ then $\alpha = -a + jb$ causes
 $|H(\omega)| = 1 \forall \omega$. This can easily be verified in Matlab
 for specific values of β .

7. Mitra 4.69

The frequency response is given by:

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[0]e^{-j2\omega} = h[0](1 + e^{-j2\omega}) + h[1]e^{-j\omega} \\ = e^{-j\omega}(2h[0]\cos(\omega) + h[1]).$$

We require that: $|H(e^{j0.2})| = 2h[0]\cos(0.2) + h[1] = 0$, $|H(e^{j0.5})| = 2h[0]\cos(0.5) + h[1] = 1$.

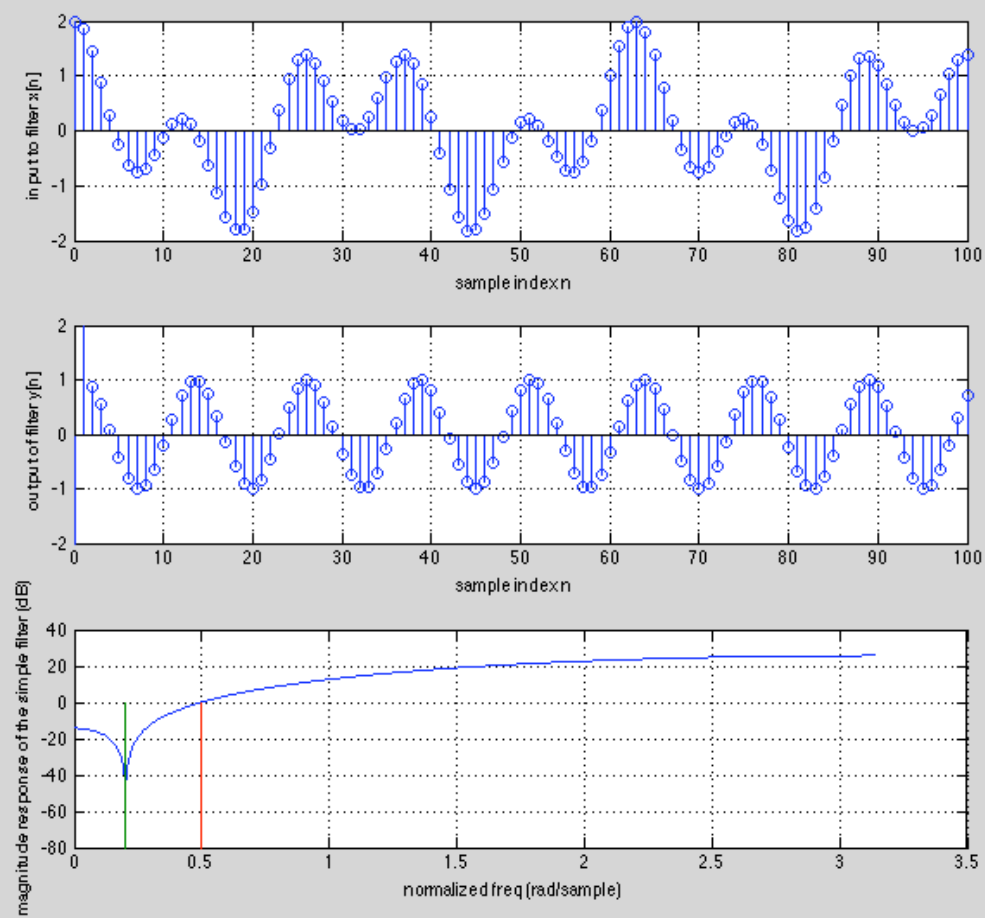
Solving these two equations we get $h[0] = -4.8788$ and $h[1] = 9.5631$.

The code for this is similar to what I demonstrated in lecture.

```
% Matlab verification of Mitra 4.69
% DRB 24-Jan-2012
```

```
omega0 = 0.2;
omega1 = 0.5;
c0 = 1;
c1 = 1;
n = 0:100;
x = c0*cos(omega0*n)+c1*cos(omega1*n);
alpha0 = 1/(2*(cos(omega1)-cos(omega0)));
alpha1 = 1-2*alpha0*cos(omega1);
h = [alpha0 alpha1 alpha0];
y = filter(h,1,x);
subplot(3,1,1)
stem(n,x);
xlabel('sample index n');
ylabel('input to filter x[n]');
grid on
subplot(3,1,2)
stem(n,y);
axis([0 max(n) -2 2]);
xlabel('sample index n');
ylabel('output of filter y[n]');
grid on
subplot(3,1,3)
[h,w]=freqz(h,1,1024);
plot(w,10*log10(abs(h).^2),[omega0 omega0],[-80 0],[omega1 omega1],[-80 0]);
xlabel('normalized freq (rad/sample)');
ylabel('magnitude response of the simple filter (dB)');
grid on
```

The results are shown on the following page.

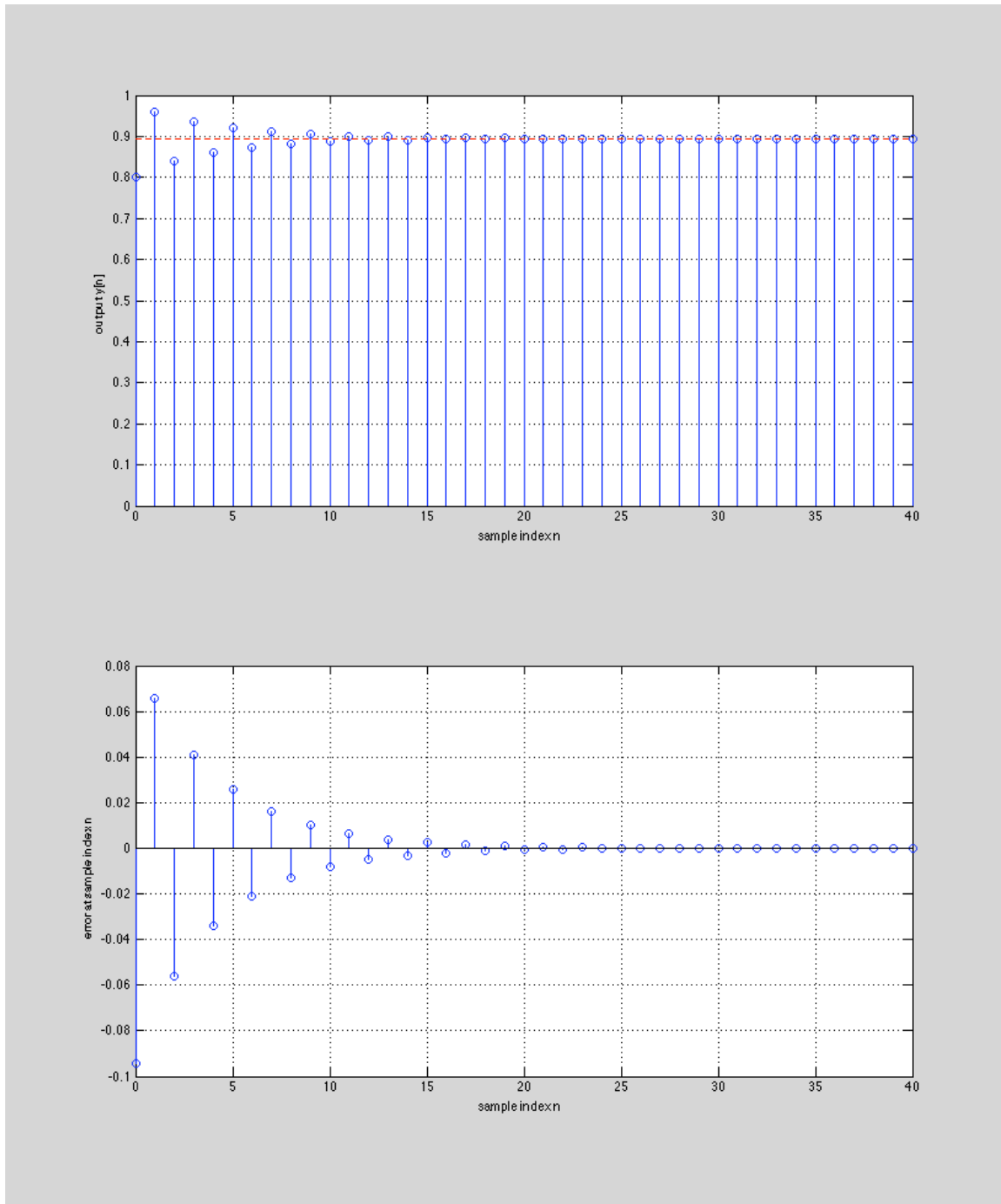


8. Mitra M4.3

Note you can't use the filter command here because this is a nonlinear system.

```
% Solution to Mitra M4.3
% DRB 24-Jan-2012
% Implementation of  $y[n] = x[n] - y^2[n-1] + y[n-1]$ 
% with  $y[-1] = 1$  and  $x[n] = \alpha u[n]$ 
% As  $n \rightarrow \infty$ ,  $y[n] \rightarrow \sqrt{\alpha}$ 
% =====
% User parameters
% =====
N = 40;           % largest sample index
alpha = 0.8;      % number to compute square root of
% =====
n = 0:N;         % sample indices
y = zeros(1,N+1); % pre-allocate output array
y(1) = alpha-1+1; % compute  $y[0]$  given  $y[-1] = 1$ 
for i=2:N+1,
    % note  $n = i-1$ 
    y(i) = alpha-y(i-1)^2+y(i-1); % compute  $y[i]$ 
end
subplot(2,1,1)
stem(n,y);
xlabel('sample index n');
ylabel('output  $y[n]$ ');
grid on
hold on
plot([0 N],[sqrt(alpha) sqrt(alpha)],'r--');
hold off
subplot(2,1,2)
stem(n,y-sqrt(alpha));
xlabel('sample index n');
ylabel('error at sample index n');
grid on
```

The results for $\alpha=0.8$ are shown on the following page.



If you want to compute the square root of a value of α greater than 1, you can pre-divide the number by a larger number with a known square root. For example, suppose you want to compute the square root of $\alpha=5$. You pre-divide 5 by 9 (which has a known square root of 3) to get $\alpha' = 5/9 < 1$. Now run the program to compute the square root of $\alpha' = 0.7454$. Then $\text{sqrt}(\alpha) = 3 \cdot 0.7454 = 2.2361$, which is the correct answer for $\text{sqrt}(5)$.