

Digital Signal Processing

Impulse Invariance vs. Bilinear Transform

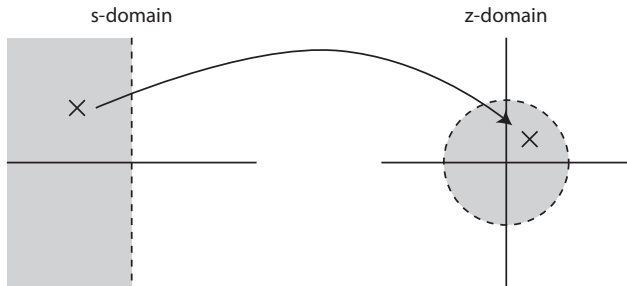
D. Richard Brown III

Properties of Impulse Invariance and the Bilinear Transform

Property 1: Points on the imaginary axis in the s -domain are mapped to points on the unit circle in the z -domain.

Property 2: Stable causal CT transfer functions are transformed to stable causal DT transfer functions.

Recall that stable causal CT transfer functions have poles in the LHP.



This doesn't say anything about the zeros, however.

Minimum Phase Systems

A minimum phase CT system is causal and has all poles and zeros in the LHP.

Does the impulse invariance method or the bilinear transform preserve this minimum phase property?

As an example, consider

$$H_c(s) = \frac{(s + 20)^2}{(s + 1)(s + 2)}$$

Using a sampling period $T_d = 1$, we can compute (or use MATLAB functions `impinvar` and `bilinear`)

$$H(z) = \begin{cases} \frac{38+69.8337z-1+0.0498z^{-2}}{1-0.5032z^{-1}+0.0498z^{-2}} & \text{via impulse invariance} \\ \frac{40.3333+66z^{-1}+27}{1-0.3333z^{-1}} & \text{via bilinear transform} \end{cases}$$

Note that the impulse invariant result is not minimum phase since it has a zero with magnitude 1.8370.

It can be shown that, in general, the **bilinear transform preserves minimum phase**.

All-Pass Systems

Suppose we have an all-pass CT filter and we wish to convert this filter to DT via impulse invariance or a bilinear transform. Does the impulse invariance method or the bilinear transform preserve the all-pass property?

For impulse invariance, we have

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c \left(j \left(\frac{\omega}{T_d} + \frac{2\pi k}{T_d} \right) \right)$$

If we have an all-pass filter with $H_c(j\Omega) = c$ for all Ω , then $H(e^{j\omega})$ doesn't even exist. In general, the aliasing inherent in the impulse invariant design method makes it inappropriate for all-pass filter design.

For the bilinear transform, there is no aliasing and the frequency response between $H(j\Omega)$ and $H(e^{j\omega})$ is mapped via $\omega = 2 \tan^{-1}(\Omega T_d/2)$. Hence the **bilinear transform preserves all-pass filter responses**.

Cascaded Systems

Suppose we have $H_c(s) = H_{c1}(s)H_{c2}(s)$ and the associated discrete-time filters $H(z)$, $H_1(z)$, and $H_2(z)$ obtained from the continuous-time filters via impulse invariance or the bilinear transform.

Does $H(z) = H_1(z)H_2(z)$ for the impulse invariance method or the bilinear transform? The answer is clearly “yes” for the bilinear transform since it just maps $s = \frac{2}{T_d} \frac{1+z^{-1}}{1-z^{-1}}$.

For impulse invariance, as an example, consider

$$H_{c1}(s) = H_{c2}(s) = \frac{s}{s+1}.$$

Using $T_d = 1$, we can compute the impulse invariant systems

$$H_1(z) = H_2(z) = \frac{-0.3679z^{-1}}{1 - 0.3679z^{-1}}$$

hence $H(z) = \left(\frac{-0.3679z^{-1}}{1 - 0.3679z^{-1}} \right)^2 = \frac{0.1354z^{-2}}{1 - 0.7358z^{-1} + 0.1354z^{-2}}$. But directly computing the impulse invariant system for $H(s) = \frac{s^2}{s^2 + 2s + 1}$ yields $H(z) = \frac{-1 + 0.3679z^{-1} + 0.1353z^{-2}}{1 - 0.7358z^{-1} + 0.1354z^{-2}}$ which is different. Hence the **bilinear transform preserves cascaded filter responses**.

Parallel Systems

Suppose we have $H_c(s) = H_{c1}(s) + H_{c2}(s)$ and the associated discrete-time filters $H(z)$, $H_1(z)$, and $H_2(z)$ obtained from the continuous-time filters via impulse invariance or the bilinear transform.

Does $H(z) = H_1(z) + H_2(z)$ for the impulse invariance method or the bilinear transform? The answer is clearly “yes” for the bilinear transform since it just maps $s = \frac{2}{T_d} \frac{1+z^{-1}}{1-z^{-1}}$.

For impulse invariance, we have

$$\begin{aligned} H(e^{j\omega}) &= \sum_{k=-\infty}^{\infty} H_c \left(j \left(\frac{\omega}{T_d} + \frac{2\pi k}{T_d} \right) \right) \\ &= \sum_{k=-\infty}^{\infty} H_{c1} \left(j \left(\frac{\omega}{T_d} + \frac{2\pi k}{T_d} \right) \right) + \sum_{k=-\infty}^{\infty} H_{c2} \left(j \left(\frac{\omega}{T_d} + \frac{2\pi k}{T_d} \right) \right) \\ &= H_1(e^{j\omega}) + H_2(e^{j\omega}) \end{aligned}$$

hence $h[n] = h_1[n] + h_2[n]$ and $H(z) = H_1(z) + H_2(z)$. Hence both the **impulse invariant method and the bilinear transform preserve parallel filter responses.**

Preserving DC Gain

The DC gain of a continuous-time filter is given as $H_c(j\Omega)$ with $\Omega = 0$. The DC gain of a discrete-time filter is given as $H(e^{j\omega})$ with $\omega = 0$.

Does $H_c(j\Omega)|_{\Omega=0} = H(e^{j\omega})|_{\omega=0}$ for the impulse invariance method or the bilinear transform?

Consider our previous example

$$H_c(s) = \frac{s}{s+1} \quad \Rightarrow \quad H_c(j) = \frac{j\Omega}{j\Omega+1}$$

which clearly has a DC gain of zero. Using $T_d = 1$, we can compute the impulse invariant system

$$H(z) = \frac{-0.3679z^{-1}}{1 - 0.3679z^{-1}} \quad \Rightarrow \quad H(e^{j\omega}) = \frac{-0.3679e^{-j\omega}}{1 - 0.3679e^{-j\omega}}$$

which does not have a DC gain of zero. This is due to aliasing in the impulse invariant method.

For the bilinear transform, we have the frequency mapping $\omega = 2 \tan^{-1}(\Omega T_d/2)$. When $\Omega = 0$, we have $\omega = 0$. Hence the **bilinear transform preserves the DC gain**.