Digital Signal Processing
Frequency Transformations of CT Lowpass Filters

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General Filter Design Procedure

The bilinear transform **lowpass filter** design procedure is straightforward:

1. **discrete-time filter specifications**
   - Prewarp DT frequency specifications to CT
   - Design CT lowpass filter
   - Bilinear transform

If you want a different type of filter, e.g. bandpass, there are two options:

1. **discrete-time filter specifications**
   - Prewarp DT frequency specifications to CT
   - Convert CT prototype lowpass filter to desired filter type via spectral transformation
   - Convert DT prototype lowpass filter to desired filter type via spectral transformation

H(z)
Frequency Transformation in the Analog Domain

Suppose you have a “prototype” continuous time lowpass filter denoted as $H_P(s)$ with passband and stopband frequencies $\Omega_p$ and $\Omega_s$, respectively. This filter can be transformed to another type of filter $H_T(s)$ by substituting $s \rightarrow F(s)$.

<table>
<thead>
<tr>
<th>type</th>
<th>TF substitution</th>
</tr>
</thead>
<tbody>
<tr>
<td>lowpass→lowpass</td>
<td>$s \rightarrow \frac{\Omega_p}{\hat{\Omega}_p} s$</td>
</tr>
<tr>
<td>lowpass→highpass</td>
<td>$s \rightarrow \frac{\Omega_p \hat{\Omega}_p}{s}$</td>
</tr>
<tr>
<td>lowpass→bandpass</td>
<td>$s \rightarrow \Omega_p \frac{s^2 + \Omega_0^2}{s(\hat{\Omega}<em>{p2} - \hat{\Omega}</em>{p1})}$</td>
</tr>
<tr>
<td>lowpass→bandstop</td>
<td>$s \rightarrow \Omega_s \frac{s(\hat{\Omega}<em>{s2} - \hat{\Omega}</em>{s1})}{s^2 + \hat{\Omega}_0^2}$</td>
</tr>
</tbody>
</table>

Note that $\hat{\Omega}_0 = \sqrt{\hat{\Omega}_{p1} \hat{\Omega}_{p2}} = \sqrt{\hat{\Omega}_{s1} \hat{\Omega}_{s2}}$ is the geometric center frequency of the passband/stopband for bandpass and bandstop filters.
Lowpass to Lowpass Transformation Example

The simplest case is a lowpass to lowpass transformation. This transformation just proportionally changes all of the relevant frequencies of the prototype LPF.

As an example, suppose our prototype LPF is a first-order Butterworth LPF with

\[ H_P(s) = \frac{1}{1 + \frac{s}{\Omega_c}} = \frac{\Omega_c}{\Omega_c + s} \]

If we perform the substitution \( s \rightarrow \frac{\Omega_p}{\hat{\Omega}_p} s \), we get

\[ H_T(s) = \frac{\Omega_c}{\Omega_c + \frac{\Omega_p}{\hat{\Omega}_p} s} = \frac{\hat{\Omega}_p \Omega_c}{\hat{\Omega}_p \Omega_c + s} = \hat{\Omega}_c + s \]

Observe that the cutoff frequency has been scaled so that \( \Omega_c \rightarrow \frac{\hat{\Omega}_p}{\hat{\Omega}_p} \Omega_c = \hat{\Omega}_c \).

This correspondingly scales the passband frequency \( \Omega_p \rightarrow \frac{\hat{\Omega}_p}{\hat{\Omega}_p} \Omega_p = \hat{\Omega}_p \) and stopband frequency \( \Omega_s \rightarrow \frac{\hat{\Omega}_p}{\hat{\Omega}_p} \Omega_s = \hat{\Omega}_s \).
Lowpass to Lowpass Transformation Example: \( \hat{\Omega}_p = 2 \)

![Graph showing magnitude response in dB vs frequency Ω for prototype and transformed LPFs.](image-url)
Lowpass to Highpass Transformation

For lowpass to highpass transformations, we use

\[ s \rightarrow \frac{\Omega_p \hat{\Omega}_p}{s} \]

Substituting \( s = j\Omega \) on the lefthand side and \( s = j\hat{\Omega} \) on the righthand side, we can relate the frequencies of the prototype and transformed systems as

\[ \Omega = -\frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}} \]

Given \( \Omega_p \) and \( \Omega_s \) of the prototype filter, we can pick our desired value of \( \hat{\Omega}_p \) for our transformed highpass filter and then compute the stopband frequency edge

\[ \hat{\Omega}_s = -\frac{\Omega_p \hat{\Omega}_p}{\Omega_s} \]

We assume a symmetric magnitude response, so the minus signs can be ignored.
First order LPF\(\rightarrow\)HPF Example: \(\Omega_p = 5, \hat{\Omega}_p = 80\)
Lowpass to Bandpass Transformation

For lowpass to bandpass transformations, we use

\[ s \rightarrow \Omega_p \frac{s^2 + \hat{\Omega}_0^2}{s (\hat{\Omega}_{p2} - \hat{\Omega}_{p1})} \]

where \( \hat{\Omega}_0 = \sqrt{\hat{\Omega}_{p1}\hat{\Omega}_{p2}} = \sqrt{\hat{\Omega}_{s1}\hat{\Omega}_{s2}} \) is the geometric center frequency of the passband/stopband. We can relate the frequencies of the prototype and transformed systems as

\[ \Omega = -\Omega_p \frac{\hat{\Omega}_0^2 - \hat{\Omega}_2}{\hat{\Omega} (\hat{\Omega}_{p2} - \hat{\Omega}_{p1})} \]

Given \( \Omega_p \) and \( \Omega_s \) of the prototype filter, we can pick \( \hat{\Omega}_{p1} \) and \( \hat{\Omega}_{p2} \) for our transformed bandpass filter and then compute the stopband frequencies

\[ \hat{\Omega}_{s1,2} = \frac{\Omega_s B}{\Omega_p} \pm \sqrt{\left(\frac{\Omega_s B}{\Omega_p}\right)^2 + 4\hat{\Omega}_0^2} \]

where \( B = \hat{\Omega}_{p2} - \hat{\Omega}_{p1} \).
First order LPF $\rightarrow$ HPF Ex.: $\Omega_p = 5, \hat{\Omega}_{p1} = 40, \hat{\Omega}_{p2} = 60$
Reverse Mappings for Bandpass and Bandstop Filters

The reverse mapping of band edges for BPF and BSF to a prototype LPF, i.e., \( \{\hat{\Omega}_p, \Omega_s\} \rightarrow \{\Omega_p, \Omega_s\} \), requires some special care. We need the geometric center frequency of the passband to be identical to the geometric center frequency of the stop band edged, i.e., we need \( \hat{\Omega}_p \hat{\Omega}_s = \hat{\Omega}_s \hat{\Omega}_p \).

For a bandpass filter, if \( \hat{\Omega}_p \hat{\Omega}_s > \hat{\Omega}_s \hat{\Omega}_s \) we can get the desired equality by:

- Increasing \( \hat{\Omega}_s \) shortening the left transition band (ok).
- Decreasing \( \hat{\Omega}_p \) shortening the left transition band (ok).
- Increasing \( \hat{\Omega}_s \) lengthening the right transition band (not ok).
- Decreasing \( \hat{\Omega}_s \) lengthening the right transition band (not ok).

You can make similar statements for the case when \( \hat{\Omega}_p \hat{\Omega}_s < \hat{\Omega}_s \hat{\Omega}_s \) and for the same cases for the BSF. The key is that the new filter specs must be more stringent than the old filter specs.