Digital Signal Processing
Frequency Transformations of CT Lowpass Filters

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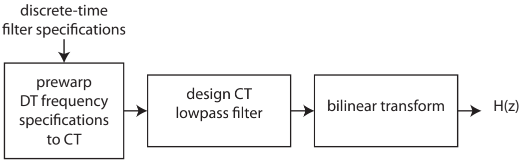
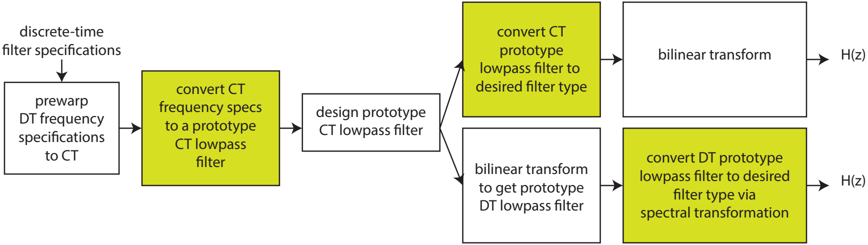
General Filter Design Procedure

The bilinear transform **lowpass filter** design procedure is straightforward:



If you want a different type of filter, e.g. bandpass, there are two options:



Frequency Transformation in the Analog Domain

Suppose you have a "prototype" continuous time lowpass filter denoted as $H_P(s)$ with passband and stopband frequencies Ω_p and Ω_s, respectively. This filter can be transformed to another type of filter $H_T(s)$ by substituting $s \to F(s)$.

type	TF substitution
lowpass→lowpass	$s \to \frac{\Omega_p}{\hat{\Omega}_p} s$
lowpass→highpass	$s \to \frac{\Omega_p \hat{\Omega}_p}{s}$
lowpass→bandpass	$s \to \Omega_p \frac{s^2+\hat{\Omega}_0^2}{s\left(\hat{\Omega}_{p2}-\hat{\Omega}_{p1}\right)}$
lowpass→bandstop	$s \to \Omega_s \frac{s\left(\hat{\Omega}_{s2}-\hat{\Omega}_{s1}\right)}{s^2+\hat{\Omega}_0^2}$

Note that $\hat{\Omega}_0 = \sqrt{\hat{\Omega}_{p1}\hat{\Omega}_{p2}} = \sqrt{\hat{\Omega}_{s1}\hat{\Omega}_{s2}}$ is the geometric center frequency of the passband/stopband for bandpass and bandstop filters.

Lowpass to Lowpass Transformation Example

The simplest case is a lowpass to lowpass transformation. This transformation just proportionally changes all of the relevant frequencies of the prototype LPF.

As an example, suppose our prototype LPF is a first-order Butterworth LPF with

$$H_P(s) = \frac{1}{1+\frac{s}{\Omega_c}} = \frac{\Omega_c}{\Omega_c + s}$$

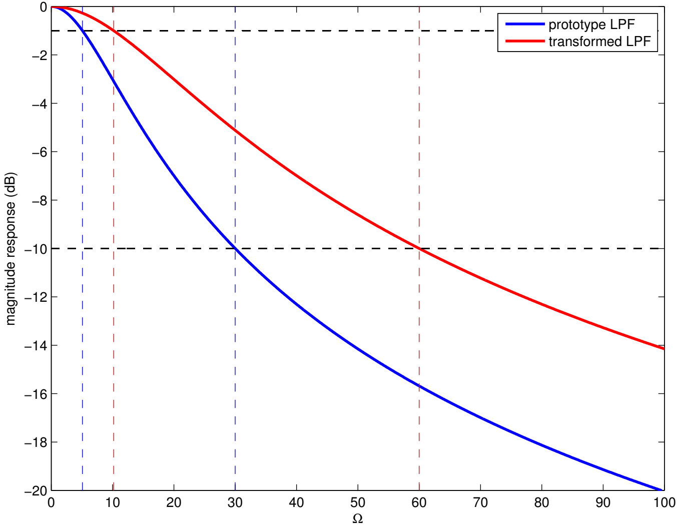
If we perform the substitution $s \to \frac{\Omega_p}{\hat{\Omega}_p} s$, we get

$$H_T(s) = \frac{\Omega_c}{\Omega_c + \frac{\Omega_p}{\hat{\Omega}_p} s} = \frac{\frac{\hat{\Omega}_p}{\Omega_p}\Omega_c}{\frac{\hat{\Omega}_p}{\Omega_p}\Omega_c + s} = \frac{\hat{\Omega}_c}{\hat{\Omega}_c + s}$$

Observe that the cutoff frequency has been scaled so that $\Omega_c \to \frac{\hat{\Omega}_p}{\Omega_p}\Omega_c = \hat{\Omega}_c$.

This correspondingly scales the passband frequency $\Omega_p \to \frac{\hat{\Omega}_p}{\Omega_p}\Omega_p = \hat{\Omega}_p$ and stopband frequency $\Omega_s \to \frac{\hat{\Omega}_p}{\Omega_p}\Omega_s = \hat{\Omega}_s$.

Lowpass to Lowpass Transformation Example: $\frac{\hat{\Omega}_p}{\Omega_p} = 2$



Lowpass to Highpass Transformation

For lowpass to highpass transformations, we use

$$s \to \frac{\Omega_p \hat{\Omega}_p}{s}$$

Substituting $s = j\Omega$ on the lefthand side and $s = j\hat{\Omega}$ on the righthand side, we can relate the frequencies of the prototype and transformed systems as

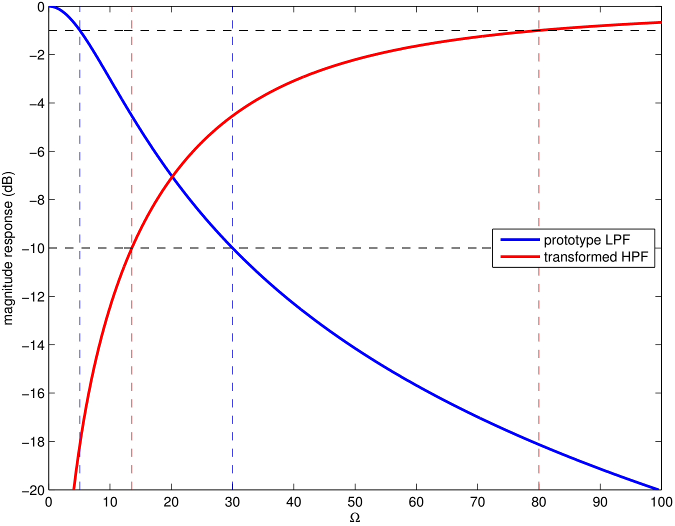
$$\Omega = -\frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}}$$

Given Ω_p and Ω_s of the prototype filter, we can pick our desired value of $\hat{\Omega}_p$ for our transformed highpass filter and then compute the stopband frequency edge

$$\hat{\Omega}_s = -\frac{\Omega_p \hat{\Omega}_p}{\Omega_s}.$$

We assume a symmetric magnitude response, so the minus signs can be ignored.

First order LPF→HPF Example: $\Omega_p = 5$, $\hat{\Omega}_p = 80$



Lowpass to Bandpass Transformation

For lowpass to bandpass transformations, we use

$$s \to \Omega_p \frac{s^2 + \hat{\Omega}_0^2}{s\left(\hat{\Omega}_{p2} - \hat{\Omega}_{p1}\right)}$$

where $\hat{\Omega}_0 = \sqrt{\hat{\Omega}_{p1}\hat{\Omega}_{p2}} = \sqrt{\hat{\Omega}_{s1}\hat{\Omega}_{s2}}$ is the geometric center frequency of the passband/stopband. We can relate the frequencies of the prototype and transformed systems as

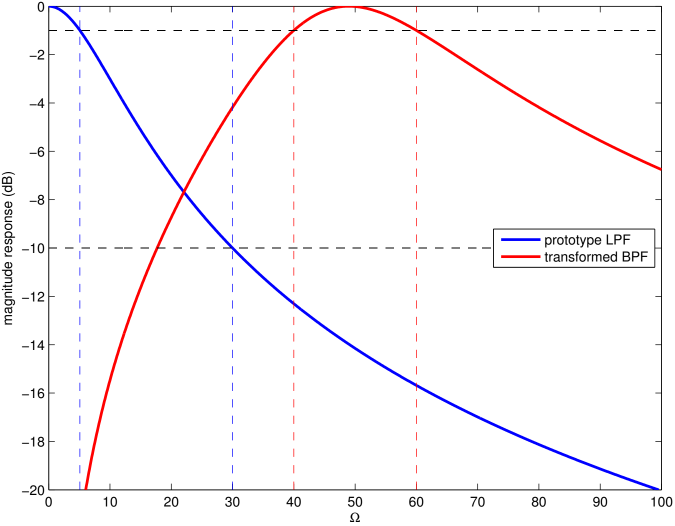
$$\Omega = -\Omega_p \frac{\hat{\Omega}_0^2 - \hat{\Omega}^2}{\hat{\Omega}\left(\hat{\Omega}_{p2} - \hat{\Omega}_{p1}\right)}$$

Given Ω_p and Ω_s of the prototype filter, we can pick $\hat{\Omega}_{p1}$ and $\hat{\Omega}_{p2}$ for our transformed bandpass filter and then compute the stopband frequencies

$$\hat{\Omega}_{s1,2} = \frac{\frac{\Omega_s B}{\Omega_p} \pm \sqrt{\left(\frac{\Omega_s B}{\Omega_p}\right)^2 + 4\hat{\Omega}_0^2}}{2}$$

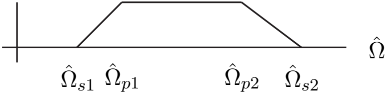
where $B = \hat{\Omega}_{p2} - \hat{\Omega}_{p1}$.

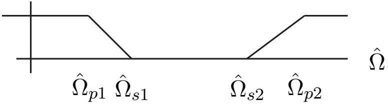
First order LPF→HPF Ex.: $\Omega_p = 5$, $\hat{\Omega}_{p1} = 40$ $\hat{\Omega}_{p2} = 60$



Reverse Mappings for Bandpass and Bandstop Filters

The reverse mapping of band edges for BPF and BSF to a prototype LPF, i.e., $\{\hat{\Omega}_{p1}, \hat{\Omega}_{p2}, \hat{\Omega}_{s1}, \hat{\Omega}_{s1}\} \rightarrow \{\Omega_p, \Omega_s\}$, requires some special care. We need the geometric center frequency of the passband to be identical to the geometric center frequency of the stop band edged, i.e., we need $\hat{\Omega}_0^2 = \hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$.





For a bandpass filter, if $\hat{\Omega}_{p1}\hat{\Omega}_{p2} > \hat{\Omega}_{s1}\hat{\Omega}_{s2}$ we can get the desired equality by:

- Increasing $\hat{\Omega}_{s1}$ shortening the left transition band (ok).
- Decreasing $\hat{\Omega}_{p1}$ shortening the left transition band (ok).
- Increasing $\hat{\Omega}_{s2}$ lengthening the right transition band (not ok).
- Decreasing $\hat{\Omega}_{p2}$ lengthening the right transition band (not ok).

You can make similar statements for the case when $\hat{\Omega}_{p1}\hat{\Omega}_{p2} < \hat{\Omega}_{s1}\hat{\Omega}_{s2}$ and for the same cases for the BSF. The key is that the new filter specs must be more stringent than the old filter specs.