Digital Signal Processing
Multirate Filter Banks

D. Richard Brown III
Multirate Filter Banks and Applications

Example of a two-channel analysis and synthesis multirate filter bank:

For example: $H_0$ is a LPF and $H_1$ is HPF.

Example applications:

- Oversampling in digital audio systems.
- Subband coding of speech and image signals.
- Encryption/security.

Conditions for Perfect Reconstruction

In the absence of any processing between the down/up-samplers, under what conditions on $H_0$, $H_1$, $G_0$, and $G_1$ will we have $y[n] = x[n - M]$?
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Let’s see what happens when $H_0 = G_0$ are ideal lowpass filters and $H_1 = G_1$ are ideal highpass filters, all with cutoff frequency $\omega_c = \pi/2$. 
\[ H_0(e^{j\omega})X(e^{j\omega}) \]
\[ H_1(e^{j\omega})X(e^{j\omega}) \]
\[ V_0(e^{j\omega}) \]
\[ V_1(e^{j\omega}) \]
\[ Y_0(e^{j\omega}) \]
\[ Y_1(e^{j\omega}) \]
\[ Y(e^{j\omega}) \]
It turns out that we don’t need ideal lowpass/highpass filters for this idea to work. As another example, suppose

\[ h_0[n] = \frac{1}{2} (\delta[n] + \delta[n - 1]) \Leftrightarrow H_0(e^{j\omega}) = \cos(\omega/2)e^{-j\omega/2} \]

\[ h_1[n] = \frac{1}{2} (\delta[n] - \delta[n - 1]) \Leftrightarrow H_1(e^{j\omega}) = j\sin(\omega/2)e^{-j\omega/2} = H_0(e^{j(\omega-\pi)}) \]

\[ g_0[n] = \delta[n] + \delta[n - 1] \Leftrightarrow G_0(e^{j\omega}) = 2 \cos(\omega/2)e^{-j\omega/2} = 2H_0(e^{j\omega}) \]

\[ g_1[n] = -\delta[n] + \delta[n - 1] \Leftrightarrow G_1(e^{j\omega}) = -2j\sin(\omega/2)e^{-j\omega/2} = -2H_0(e^{j(\omega-\pi)}) \]
Note that the analysis stage filters and downsamples by $M = 2$. Hence, we can write

\[ V_0(e^{j\omega}) = \frac{1}{2} \left[ X(e^{j\omega/2})H_0(e^{j\omega/2}) + X(e^{j(\omega/2-\pi)})H_0(e^{j(\omega/2-\pi)}) \right] \]

\[ = e^{-j\omega/4} \left[ \frac{X(e^{j\omega/2}) \cos(\omega/4)}{2} + X(e^{j(\omega/2-\pi)})j \sin(\omega/4) \right] \]

and

\[ V_1(e^{j\omega}) = \frac{1}{2} \left[ X(e^{j\omega/2})H_1(e^{j\omega/2}) + X(e^{j(\omega/2-\pi)})H_1(e^{j(\omega/2-\pi)}) \right] \]

\[ = e^{-j\omega/4} \left[ \frac{X(e^{j\omega/2})j \sin(\omega/4)}{2} + X(e^{j(\omega/2-\pi)}) \cos(\omega/4) \right] \]
Analysis (part 2 of 2)

The synthesis stage upsamples by $L = 2$ and then filters. Generally, we have

$$Y_i(e^{j\omega}) = G_i(e^{j\omega})V_i(e^{j2\omega})$$

hence we can write

$$Y_0(e^{j\omega}) = 2 \cos(\omega/2) e^{-j\omega/2} \cdot \frac{e^{-j\omega/2}}{2} \left[ X(e^{j\omega}) \cos(\omega/2) + X(e^{j(\omega-2\pi)}) j \sin(\omega/2) \right]$$

$$= \cos(\omega/2) e^{-j\omega} X(e^{j\omega}) [\cos(\omega/2) + j \sin(\omega/2)]$$

$$= e^{-j\omega} X(e^{j\omega}) [\cos^2(\omega/2) + j \cos(\omega/2) \sin(\omega/2)]$$

and

$$Y_1(e^{j\omega}) = -2j \sin(\omega/2) e^{-j\omega/2} \cdot \frac{e^{-j\omega/2}}{2} \left[ X(e^{j\omega}) j \sin(\omega/2) + X(e^{j(\omega-2\pi)}) \cos(\omega/2) \right]$$

$$= -j \sin(\omega/2) e^{-j\omega} X(e^{j\omega}) [j \sin(\omega/2) + \cos(\omega/2)]$$

$$= e^{-j\omega} X(e^{j\omega}) [\sin^2(\omega/2) - j \cos(\omega/2) \sin(\omega/2)]$$

It follows then that

$$Y(e^{j\omega}) = Y_0(e^{j\omega}) + Y_1(e^{j\omega}) = e^{-j\omega} X(e^{j\omega})$$

which means we've recovered our original signal with a one-sample delay.
Remarks

- The filters in the previous examples satisfy the “alias cancellation condition” which requires

\[ G_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + G_1(e^{j\omega})H_1(e^{j(\omega-\pi)}) = 0 \]

- You can also use the polyphase implementation ideas we developed earlier to reorder the filtering and up/downsampling in the analysis and synthesis operations to save computation: