Digital Signal Processing
Inferring Poles and Zeros from the Magnitude Response

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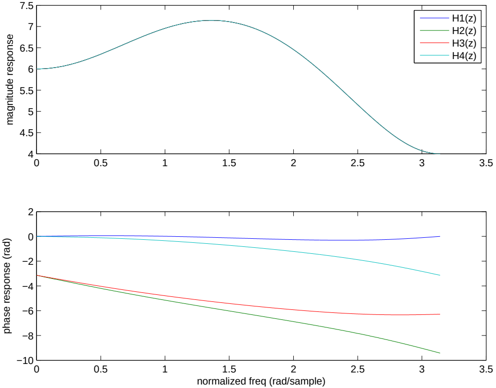
Different Systems with the Same Magnitude Response

$$H_1(z) = 6 + z^{-1} - z^{-2} \qquad H_2(z) = 1 - z^{-1} - 6z^{-2}$$
$$H_3(z) = 2 - 5z^{-1} - 3z^{-2} \qquad H_4(z) = 3 + 5z^{-1} - 2z^{-2}$$

All four of these causal stable systems have the same magnitude response but they have different phase responses.



Assumptions and Problem Setup

1. The system is described by a rational transfer function with real coefficients
$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}.$$
2. The system is causal and stable (hence, all poles are inside the unit circle).

We are given only the magnitude response $|H(e^{j\omega})|^2$ and would like to determine the poles and zeros of $H(z)$.

Spectral Factorization

Let

$$P(e^{j\omega}) = |H(e^{j\omega})|^2$$

Note that since $P(e^{j\omega})$ is a power spectrum, it is the DTFT of an autocorrelation function. Hence, it has an inverse DTFT $p[n]$ of the form

$$p[n] = h[n] * h[-n] = h[n] * h^*[-n]$$

where the second equality results from the fact that $h[n]$ is assumed to be real. Hence, we can write

$$P(z) = H(z)H^*(1/z^*)$$

with

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{(1-c_1 z^{-1})\cdots(1-c_M z^{-1})}{(1-d_1 z^{-1})\cdots(1-d_N z^{-1})}$$

and

$$H^*(1/z^*) = \left(\frac{b_0}{a_0}\right)\frac{(1-c_1^* z)\cdots(1-c_M^* z)}{(1-d_1^* z)\cdots(1-d_N^* z)}$$

Determining the Poles of H(z) from $P(z)$

Recall we know $P(z)$ but not $H(z)$ or $H^*(1/z^*)$.

We can write

$$P(z) = \left(\frac{b_0}{a_0}\right)^2 \frac{(1-c_1z^{-1})\cdots(1-c_Mz^{-1})}{(1-d_1z^{-1})\cdots(1-d_Nz^{-1})} \frac{(1-c_1^*z)\cdots(1-c_M^*z)}{(1-d_1^*z)\cdots(1-d_N^*z)}$$

Hence we see that the zeros of $P(z)$ are at $\{c_1, 1/c_1^*, \dots, c_M, 1/c_M^*\}$ and the poles of $P(z)$ are at $\{d_1, 1/d_1^*, \dots, d_N, 1/d_N^*\}$ (plus possibly some poles or zeros at zero or infinity).

Since, for any complex or real number α, $|\alpha| = \frac{1}{|1/\alpha^*|}$ and $\angle\alpha = \angle 1/\alpha^*$, we see that the poles and zeros are mirrored across the unit circle.

Under our assumption that $H(z)$ is stable and causal, we know $|d_n| \neq 1$ for all $n = 1, \dots, N$, and the poles of $H(z)$ are the N poles of $P(z)$ inside the unit circle. Hence, the poles of $H(z)$ are uniquely determined.

Nonuniqueness of the Zeros of H(z) from $P(z)$

Suppose, for example,

$$P(z) = \frac{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)}{(1-\frac{2}{3}z^{-1})(1-\frac{2}{3}z)} = \frac{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}{(1-\frac{2}{3}z^{-1})(1-\frac{3}{2}z^{-1})}$$

with $c_1 = \frac{1}{2}$ and $d_1 = 2/3$. While the pole is uniquely determined by stability and causality, we can choose either zero without affecting the magnitude response. In fact

$$H_1(z) = \frac{1-\frac{1}{2}z^{-1}}{1-\frac{2}{3}z^{-1}}$$

and

$$H_2(z) = \frac{1-2z^{-1}}{1-\frac{2}{3}z^{-1}}$$

with ROC $|z| > \frac{2}{3}$ are both stable causal systems with the same magnitude response. These systems do not have the same phase response, however.

z-Plane Example