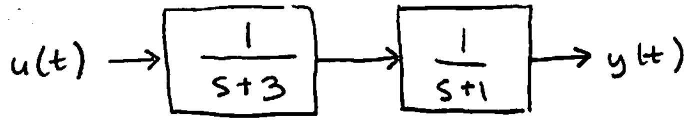
1. (a)



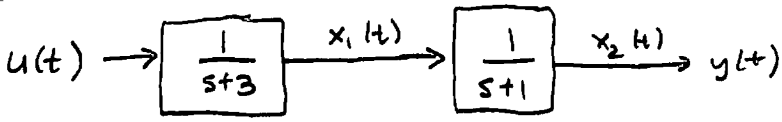

hence $\hat{y}(s) = \underbrace{\frac{1}{(s+3)(s+1)}}_{\hat{g}(s)} \hat{u}(s)$

Since $\hat{g}(s)$ is a rational function of $s$, we know this system is <u>lumped</u>.

$\hat{g}(s)$ is proper $\Leftrightarrow$ the system is <u>causal</u>

Since the system can be described by a transfer function, we know that the system must be <u>linear</u> and <u>time-invariant</u>.

(b)




the output equation is easy since $y(t) = x_2(t)$

$$\boxed{y(t) = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} u(t)}$$

From the figure $\hat{x}_2(s) = \frac{1}{s+1}\hat{x}_1(s) \Leftrightarrow \dot{x}_2(t) + x_2(t) = x_1(t)$

$$\hat{x}_1(s) = \frac{1}{s+3}\hat{u}(s) \Leftrightarrow \dot{x}_1(t) + 3x_1(t) = u(t)$$

Rearrange these results to get the state update equation

$$\boxed{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -3 & 0 \\ 1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u(t)}$$

(c) We found the transfer function in part (a), but we can also confirm that we got the answer to part (b) right by computing

$$\hat{g}(s) = C(sI-A)^{-1}B + D$$

$$sI-A = \begin{bmatrix} s+3 & 0 \\ -1 & s+1 \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{1}{(s+3)(s+1)} \begin{bmatrix} s+1 & 0 \\ 1 & s+3 \end{bmatrix}$$

so $$\hat{g}(s) = \frac{1}{(s+3)(s+1)} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ 1 & s+3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{(s+3)(s+1)}$$ ✓

(d) To find a different state space realization, we can use any of the tricks we used in our homework...

This one is easy: $\bar{A} = A^T$

$\bar{B} = C^T$

$\bar{C} = B^T$

$\bar{D} = D$

the new system is then

$$\boxed{\begin{aligned} \dot{\underline{x}}(t) &= \underbrace{\begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix}}_{\bar{A}} \underline{x}(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\bar{B}} u(t) \\ y(t) &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\bar{C}} \underline{x}(t) + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\bar{D}} u(t) \end{aligned}}$$

2. We have $\underline{x}(0)$ and we want to compute $y(t)$ given $u(t) \equiv 0$.

When we have no input, we know that

$$y(t) = C\,\Phi(t, t_0)\,\underline{x}(t_0)$$ where $t_0 = 0$ and $\underline{x}(t_0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in this problem.

$$A = \begin{bmatrix} -3 & 0 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \Phi(t,0) = e^{tA}$$ since this is a linear time-invariant system.

We need to compute $e^{tA}$...

The eigenvalues of $A$ are $\lambda_1 = -3$ and $\lambda_2 = -1$. These are distinct, so we know $A$ is diagonalizable.

To compute the eigenvectors, we need to find basis vectors for $\text{null}(A - \lambda_1 I_2)$ and $\text{null}(A - \lambda_2 I_2)$.

It can be shown that $v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \rightarrow$ check $Av_1 = \begin{bmatrix} -6 \\ 3 \end{bmatrix} = \lambda_1 v_1$ ok

and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow$ check $Av_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \lambda_2 v_2$ ok

hence $V = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$ and $V^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

Then $e^{tA} = V e^{t\Lambda} V^{-1}$ for $\Lambda = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$

computing the result...

$$e^{tA} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^{-3t} & 0 \\ e^{-t} & 2e^{-t} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2e^{-3t} & 0 \\ e^{-t} - e^{-3t} & 2e^{-t} \end{bmatrix} = \begin{bmatrix} e^{-3t} & 0 \\ \dfrac{e^{-t} - e^{-3t}}{2} & e^{-t} \end{bmatrix}$$

putting it all together

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} e^{tA} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-3t} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

hence

$$\boxed{y(t) = e^{-3t} \quad \forall t \in \mathbb{R}}$$

3.(a) By inspection, it should be clear that this system is <u>time-varying</u> due to the $k$ term in front of $y[k-1]$.

Also, this system is clearly <u>causal</u> since the current output depends only on past outputs and past inputs

The only memory needed is the previous output, hence <u>lumped</u>.

This system is <u>linear</u> (and this will be shown in part(b))

(b) Let the scalar state $x[k] = y[k]$

then $x[k+1] = y[k+1] = (k+1)\, y[k] + u[k]$

$$\boxed{\begin{aligned} x[k+1] &= \underbrace{(k+1)}_{A[k]} x[k] + \underbrace{[1]}_{B} u[k] \\ y[k] &= \underbrace{[1]}_{C} x[k] + \underbrace{[0]}_{D} u[k] \end{aligned}}$$

$\begin{bmatrix} \text{clearly} \\ \text{linear} \end{bmatrix}$

(c) When $B[\ell] \equiv 1$, we can write

$$y[k] = \Phi[k, k_0]\, x[k_0] + \sum_{\ell=k_0}^{k-1} \Phi[k, \ell+1]\, u[\ell] \qquad (*)$$

We just need an expression for $\Phi[k,j] = A[k-1]\, A[k-2] \cdots A[j]$

everything is scalar here, so this isn't too difficult

$$\Phi[k,j] = k \cdot (k-1) \cdot (k-2) \cdots (j+1) = \frac{k!}{j!} \qquad (\text{check } \Phi[k,k] = 1 \checkmark)$$

so this can just be plugged into $(*)$ for the final result.

4. (a) When $a=0$ and $b=0$, $A = I_3$ and is already diagonal

so $e^{tA} = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix}$ and $A^k = \begin{bmatrix} 1^k & 0 & 0 \\ 0 & 1^k & 0 \\ 0 & 0 & 1^k \end{bmatrix} = I_3$.

(b) Recognize that $A$ has only one distinct e-value, $\lambda_1 = 1$

The algebraic multiplicity of this e-value is $r_1 = 3$

Find the geometric multiplicity

$$\text{null}\{A - \lambda_1 I_3\} = \text{null}\left\{ \underbrace{\begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}}_{\text{this matrix only has a 2 dimensional nullspace}} \right\} = E(\lambda_1)$$

hence $A$ is not diagonalizable. Need to look at generalized eigenspace $F(\lambda_1)$

$$\text{null}\{(A - \lambda_1 I_3)^2\} = \text{null}\left\{ \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{here we have a 3-dimensional nullspace with any basis we want}} \right\} = F(\lambda_1)$$

a basis for this nullspace is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Then $V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 = V^{-1}$

and $V^{-1} A V = \Lambda + \hat{N}$

diagonal part ($\Lambda$), nilpotent part ($\hat{N}$)

hence $\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\hat{N} = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}$

Then $e^{tA} = V e^{t\Lambda} e^{t\hat{N}} V^{-1}$, we can ignore $V$ from now on since $V = I_3$.

$e^{t\Lambda}$ is easy since $\Lambda$ is diagonal

$$e^{t\Lambda} = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix} = e^t I_3$$

$e^{t\hat{N}}$ isn't too bad — directly from defn...

$$e^{t\hat{N}} = \sum_{k=0}^{\infty} \frac{\hat{N}^k t^k}{k!} = I_3 + t\begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & at \\ 0 & 1 & bt \\ 0 & 0 & 1 \end{bmatrix}$$

then $e^{tA} = e^{t\Lambda} e^{t\hat{N}} = \begin{bmatrix} e^t & 0 & ate^t \\ 0 & e^t & bte^t \\ 0 & 0 & e^t \end{bmatrix}$

c) $A^{100} = V(\Lambda + \hat{N})^{100} V^{-1}$

use binomial expansion

$$(\Lambda + \hat{N})^{100} = \sum_{m=0}^{100} \binom{100}{m} \hat{N}^m \Lambda^{100-m}$$

$$= \Lambda^{100} + 100 \cdot \hat{N}\Lambda^{99} + \text{ a bunch of zero terms}$$

since $\hat{N}^m = 0$ for $m \geq 2$.

but $\Lambda = I_3$, hence

$$A^{100} = I_3 + 100\,\hat{N}$$

$$A^{100} = \begin{bmatrix} 1 & 0 & 100a \\ 0 & 1 & 100b \\ 0 & 0 & 1 \end{bmatrix}$$