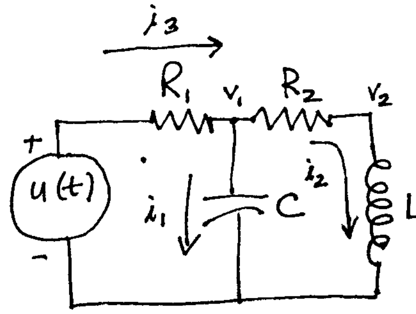
EE504 Homework Assignment #1

Solution to Problem 1a)

$$x(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$



$$\left.\begin{aligned} i_1 &= C\frac{dv_1}{dt} \\ i_1 &= i_3 - i_2 \\ i_3 &= \frac{u - v_1}{R_1} \\ i_2 &= \frac{v_1 - v_2}{R_2} \end{aligned}\right\} \Rightarrow$$

$$C\dot{v}_1 = \frac{u - v_1}{R_1} - \frac{v_1 - v_2}{R_2}$$

$$\dot{v}_1 = \frac{1}{R_1 C}(u - v_1) - \frac{1}{R_2 C}(v_1 - v_2)$$

$$\boxed{\dot{v}_1 = -\left(\frac{1}{R_1 C} + \frac{1}{R_2 C}\right)v_1 + \frac{1}{R_2 C}v_2 + \frac{1}{R_1 C}u} \quad (*)$$

$$\left.\begin{aligned} v_2 &= L\frac{di_2}{dt} \\ i_2 &= \frac{v_1 - v_2}{R_2} \end{aligned}\right\} \Rightarrow$$

$$v_2 = \frac{L}{R_2}(\dot{v}_1 - \dot{v}_2)$$

$$\frac{R_2}{L}v_2 = \dot{v}_1 - \dot{v}_2$$

$$\dot{v}_2 = \dot{v}_1 - \frac{R_2}{L}v_2$$

substitute $(*)$

$$\dot{v}_2 = -\left(\frac{1}{R_1 C} + \frac{1}{R_2 C}\right)v_1 + \frac{1}{R_2 C}v_2 + \frac{1}{R_1 C}u - \frac{R_2}{L}v_2$$

$$\boxed{\dot{v}_2 = -\left(\frac{1}{R_1 C} + \frac{1}{R_2 C}\right)v_1 + \left(\frac{1}{R_2 C} - \frac{R_2}{L}\right)v_2 + \frac{1}{R_1 C}u} \quad (**)$$

Then

$$\dot{x} = \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -\left(\frac{1}{R_1 C} + \frac{1}{R_2 C}\right) & \frac{1}{R_2 C} \\ -\left(\frac{1}{R_1 C} + \frac{1}{R_2 C}\right) & \frac{1}{R_2 C} - \frac{R_2}{L} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} \frac{1}{R_1 C} \\ \frac{1}{R_1 C} \end{bmatrix}}_{b} u$$

$$y = \frac{V_1 - V_2}{R_2} \Rightarrow y = \underbrace{\begin{bmatrix} \frac{1}{R_2} & -\frac{1}{R_2} \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} u$$

Solution to problem 2b)

$$y = i_2 = \frac{V_1 - V_2}{R_2}$$

$$V_2 = L\frac{di_2}{dt} = L\frac{dy}{dt}$$

$$\boxed{y = \frac{V_1 - L\dot{y}}{R_2}} \quad (*)$$

still need to eliminate V_1

$$i_1 = C\frac{dV_1}{dt} \quad \text{but } i_1 = i_3 - i_2 = i_3 - y$$

$$i_3 = \frac{u - V_1}{R_1}$$

hence $i_1 = \dfrac{u - V_1}{R_1} - y$

and $C\dfrac{dV_1}{dt} = \dfrac{u - V_1}{R_1} - y \Leftrightarrow \boxed{\dot{V}_1 = -\dfrac{1}{R_1 C}V_1 + \dfrac{1}{R_1 C}u - \dfrac{1}{C}y}$ $(**)$

From $(*)$: $R_2 y = V_1 - L\dot{y}$ $\quad (1)$

differentiate: $R_2 \dot{y} = \dot{V}_1 - L\ddot{y}$

substitute

$$R_2 \dot{y} = -\frac{1}{R_1 C}V_1 + \frac{1}{R_1 C}u - \frac{1}{C}y - L\ddot{y}$$

$$R_1 R_2 C \dot{y} = -V_1 + u - R_1 y - R_1 C L \ddot{y} \quad (2)$$

Add (1) and (2)

$$R_2 y + R_1 R_2 C\dot{y} = -L\dot{y} + u - R_1 y - R_1 C L\ddot{y}$$

hence

$$\boxed{R_1 C L \ddot{y} + (R_1 R_2 C + L)\dot{y} + (R_1 + R_2) y = u}$$

Solution to 1c)

We know that $\hat{g}(s) = C(SI-A)^{-1}B + D$

In this problem, $\hat{g}(s) = c(SI-A)^{-1}b$

Since A is a 2x2 matrix here, using the known formular for the 2x2 matrix inverse

Then we get

$$\hat{g}(s) = \frac{\frac{1}{R_1 C L}}{S^2 + \left(\frac{1}{R_1 C} + \frac{R_2}{L}\right)S + \left(\frac{R_2}{R_1 C L} + \frac{1}{CL}\right)}$$

PS: Problem 5 is the general case of 2x2 matrix transfer function

Solution to 1d)

Lumped, causal, linear, time invariant.

- Lumped because system has memory but number of states is finite (two).
- Causal because output depends only on present and past inputs — no future inputs.
- Linear because the system satisfies linearity properties: homogeneity and additivity.
- Time Invariant because delaying input by T leads to same output, delayed by T (with same initial conditions).

Solution to 2.

$$y(t) = \begin{cases} u^2(t)/u(t-1) & \text{if } u(t-1) \neq 0 \\ 0 & \text{if } u(t-1) = 0 \end{cases}$$

- Test homogeneity:

  let the state $x(t_0)$ be the values of $u(t)$ for $t \in [t_0-1, t_0)$. (note that the state has infinite dimensions, hence this system is distributed).

  We know that

$$\left.\begin{array}{l} x(t_0) \\ u(t) \quad t \geq t_0 \end{array}\right\} \to y(t) = \begin{cases} u^2(t)/u(t-1) & \text{if } u(t-1) \neq 0 \\ 0 & \text{if } u(t-1) = 0 \end{cases}$$

then

$$\left.\begin{array}{l} \alpha x(t_0) \\ \alpha u(t) \quad t \geq t_0 \end{array}\right\} \to \begin{cases} \alpha^2 u^2(t)/\alpha u(t-1) & \text{if } u(t-1) \neq 0 \\ 0 & \text{if } u(t-1) = 0 \end{cases}$$

but

$$\alpha^2 u^2(t)/\alpha u(t-1) = \alpha y(t) \quad \text{if } u(t-1) \neq 0$$

and

$$0 = \alpha y(t) \quad \text{if } u(t-1) = 0$$

so

$$\left.\begin{array}{l} \alpha x(t_0) \\ \alpha u(t) \quad t \geq t_0 \end{array}\right\} \to \alpha y(t)$$

and homogeneity is satisfied.

- Test additivity:

$$\left.\begin{array}{l} x_1(t_0) \\ u_1(t) \quad t \geq t_0 \end{array}\right\} \to y_1(t) = \begin{cases} u_1^2(t)/u_1(t-1) & \text{if } u_1(t-1) \neq 0 \\ 0 & \text{if } u_1(t-1) = 0 \end{cases}$$

$$\left.\begin{array}{l} x_2(t_0) \\ u_2(t) \quad t \geq t_0 \end{array}\right\} \to y_2(t) = \begin{cases} u_2^2(t)/u_2(t-1) & \text{if } u_2(t-1) \neq 0 \\ 0 & \text{if } u_2(t-1) = 0 \end{cases}$$

hence

$$\left.\begin{array}{l} x_1(t_0)+x_2(t_0) \\ u_1(t)+u_2(t) \quad t \ge t_0 \end{array}\right\} \rightarrow \begin{cases} [u_1(t)+u_2(t)]^2 / [u_1(t-1)+u_2(t-1)] & \text{if } u_1(t-1)+u_2(t-1) \ne 0 \\ 0 & \text{if } u_1(t-1)+u_2(t-1)=0 \end{cases}$$

but $y_1(t)+y_2(t) =$

$$\begin{cases} \dfrac{u_1^2(t)}{u_1(t-1)} + \dfrac{u_2^2(t)}{u_2(t-1)} & \text{if } u_1(t-1)\ne 0 \text{ and } u_2(t-1)\ne 0 \\ \dfrac{u_1^2(t)}{u_1(t-1)} & \text{if } u_1(t-1)\ne 0 \text{ and } u_2(t-1)=0 \\ \dfrac{u_2^2(t)}{u_2(t-1)} & \text{if } u_1(t-1)=0 \text{ and } u_2(t-1)\ne 0 \\ 0 & \text{if } u_1(t-1)=0 \text{ and } u_2(t-1)=0 \end{cases}$$

it is clear now that

$$\left.\begin{array}{l} x_1(t_0)+x_2(t_0) \\ u_1(t)+u_2(t) \quad t \ge t_0 \end{array}\right\} \not\rightarrow y_1(t)+y_2(t)$$

hence additivity condition fails.

[Note that if $u(t)=0$ for all t, additivity passes! However, additivity must pass for any arbitrary choice of $u(t)$.]

Solution to 3a

$$y(t) = \min(u_1(t), u_2(t))$$

- memoryless since output only depends on present inputs
- causal - all memoryless systems are causal
- Nonlinear:

  Pf: let $u_1(t) = -1$, $u_2(t) = 5$ for some value of t

  $y(t) = \min(u_1(t), u_2(t)) = \min(-1, 5) = -1$

  Now let $\alpha = -2$

  $\min(\alpha u_1(t), \alpha u_2(t)) = \min(2, -10) = -10$

  but $\alpha y(t) = 2 \neq \min(\alpha u_1(t), \alpha u_2(t))$

  hence homogeneity fails $\Rightarrow$ nonlinear.

- Time invariant

  $$\min(u_1(t+T), u_2(t+T)) = y(t+T)$$

Solution to 3b

$$y(k) = \frac{1}{1-\lambda} \sum_{m=0}^{\infty} \lambda^m u(k-m+1)$$

- distributed - requires an infinite number of states
- non causal - $y(k) = \frac{1}{1-\lambda}[u(k+1) + u(k) + \dots]$

  $\uparrow$ future input $\Rightarrow$ noncausal

- Linear. Clearly satisfies homogeneity and we will check additivity:

Let the state $x(K_o)$ be the infinite number of points $u(K_o - n)$ for $n = 1, 2, \dots$

Then

$$\left.\begin{matrix} x_1(k_o) \\ u_1(k) \quad k \geq k_o \end{matrix}\right\} \rightarrow y_1(k) = \frac{1}{1-\lambda}\sum_{m=0}^{\infty}\lambda^m u_1(k-m+1)$$

$$\left.\begin{matrix} x_2(k_o) \\ u_2(k) \quad k \geq k_o \end{matrix}\right\} \rightarrow y_2(k) = \frac{1}{1-\lambda}\sum_{m=0}^{\infty}\lambda^m u_2(k-m+1)$$

and

$$\left.\begin{matrix} x_1(k_o) + x_2(k_o) \\ u_1(k) + u_2(k) \quad k \geq k_o \end{matrix}\right\} \rightarrow \frac{1}{1-\lambda}\sum_{m=0}^{\infty}\lambda^m \left[u_1(k-m+1) + u_2(k-m+1)\right]$$

but $\frac{1}{1-\lambda}\sum_{m=0}^{\infty}\lambda^m \left[u_1(k-m+1) + u_2(k-m+1)\right]$

$$= \frac{1}{1-\lambda}\sum_{m=0}^{\infty}\lambda^m u_1(k-m+1) + \frac{1}{1-\lambda}\sum_{m=0}^{\infty}\lambda^m u_2(k-m+1)$$

$= y_1(k) + y_2(k)$ hence additivity is satisfied.

- Time invariant since $\frac{1}{1-\lambda}\sum_{m=0}^{\infty}\lambda^m u(k-m+1-N) = y(k-N)$
- If $u(k) = 1$ for all k then

$$y(k) = \frac{1}{1-\lambda}\sum_{m=0}^{\infty}\lambda^m$$

but $\sum_{m=0}^{\infty}\lambda^m = \frac{1}{1-\lambda}$ for $0 < \lambda < 1$ (geometric series)

hence $y(k) = \frac{1}{(1-\lambda)^2}$ for all k.

Solution to 4a

$$y(k) = \frac{1}{N}\sum_{n=0}^{N-1} u(k-n) \quad , \; N=4$$

$$y(k) = \frac{1}{4}\left[u(k) + u(k-1) + u(k-2) + u(k-3)\right] \quad (*)$$

$$x(k) = \begin{bmatrix} u(k-1) \\ u(k-2) \\ u(k-3) \end{bmatrix} \Rightarrow x(k+1) = \begin{bmatrix} u(k) \\ u(k-1) \\ u(k-2) \end{bmatrix}$$

hence

$$x(k+1) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k)$$

and from (*)

$$y(k) = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} x(k) + \begin{bmatrix} \frac{1}{4} \end{bmatrix} u(k)$$

Solution to 4b

$$x(k) = \begin{bmatrix} u(k-1) + u(k-2) + u(k-3) \\ u(k-1) + u(k-2) \\ u(k-1) \end{bmatrix}$$

$$\Rightarrow x(k+1) = \begin{bmatrix} u(k) + u(k-1) + u(k-2) \\ u(k) + u(k-1) \\ u(k) \end{bmatrix}$$

hence

$$x(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(k)$$

and from (*)

$$y(k) = \begin{bmatrix} \frac{1}{4} & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} \frac{1}{4} \end{bmatrix} u(k)$$

Solution to 4c

From parts (a) and (b) we saw that we are able to express the same system with two different choices for the state. Clearly the state is not unique and there exist several choices for the state that will lead to the same input-output relationship. However, the choice of state does lead to unique values for A, B, C, and D and different state vectors lead to different A, B, C, and D matrices.

EE504 Homework Assignment #1

Solution to Problem 5

$$\dot{x}(t) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} x(t) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} x(t) + d\,u(t)$$

we know that $\hat{g}(s) = C(sI-A)^{-1}B + D$

first compute $(sI-A)^{-1}$ ...

$$(sI-A)^{-1} = \begin{bmatrix} s-a_{11} & -a_{12} \\ -a_{21} & s-a_{22} \end{bmatrix}^{-1} = \frac{1}{(s-a_{11})(s-a_{22}) - a_{21}a_{12}} \cdot \begin{bmatrix} s-a_{22} & a_{12} \\ a_{21} & s-a_{11} \end{bmatrix}$$

using the known formula for the 2×2 matrix inverse.

$$\underbrace{C(sI-A)^{-1}B+D}_{\hat{g}(s)} = \left(\frac{1}{(s-a_{11})(s-a_{22}) - a_{21}a_{12}} \cdot \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} s-a_{22} & a_{12} \\ a_{21} & s-a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) + d$$

just grind it out ...

$$\hat{g}(s) = \frac{c_1 b_1 (s-a_{22}) + c_1 b_2 a_{12} + c_2 b_1 a_{21} + c_2 b_2 (s-a_{11})}{(s-a_{11})(s-a_{22}) - a_{21}a_{12}} + d$$

$$\hat{g}(s) = \frac{(c_1 b_1 + c_2 b_2)s + (c_1 b_2 a_{12} + c_2 b_1 a_{21} - c_1 b_1 a_{22} - c_2 b_2 a_{11})}{s^2 + (-a_{11} - a_{22})s + (a_{11}a_{22} - a_{21}a_{12})} + d$$

just need to incorporate "d" such that $\hat{g}(s) = \dfrac{N(s)}{D(s)}$ ← s-polynomial ← s-polynomial.

$$\boxed{\hat{g}(s) = \frac{ds^2 + (c_1 b_1 + c_2 b_2 - d a_{11} - d a_{22})s + (c_1 b_2 a_{12} + c_2 b_1 a_{21} - c_1 b_1 a_{22} - c_2 b_2 a_{11} + d a_{11} a_{22} - d a_{21} a_{12})}{s^2 + (-a_{11} - a_{22})s + (a_{11}a_{22} - a_{21}a_{12})}}$$

Solution to Problem 6:

- Impulse responses for both systems are identical
- This is somewhat surprising since the systems do not appear to be similar at all, in fact they do not have the same dimension!

What is going on?

Compute the transfer function of the first system

$$\hat{g}(s) = C(sI-A)^{-1}B + D = \frac{1}{s(s+3)+2} \cdot \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{s+1}{s^2+3s+2}$$

Key step: notice that denominator $= (s+1)(s+2)$

hence $(s+1)$ terms cancel in numerator & denominator and

$$\hat{g}(s) = \frac{1}{s+2} \qquad \text{(first system)}$$

Now look at the second system:

$$\hat{g}(s) = C(sI-A)^{-1}B + D$$

$$\hat{g}(s) = 1 \cdot \frac{1}{s+2} \cdot 1 + 0 = \frac{1}{s+2}$$

Same transfer function as first system.

Two systems with the same transfer function $\Longleftrightarrow$ Two systems with the same impulse response