ECE 504 Solution to HW #3

1. Chen problem 3.11

3.11 If and only if the $n \times n$ matrix

$$[b \ Ab \ \cdots \ A^{n-1}b]$$

is nonsingular or has full row rank.

2. Chen problem 3.16

3.16 Direct verification:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-1}{\alpha_4} & \frac{-\alpha_1}{\alpha_4} & \frac{-\alpha_2}{\alpha_4} & \frac{-\alpha_3}{\alpha_4} \end{bmatrix} \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = I_4$$

This shows the inverse. Note that if $\alpha_4 = 0$, then $\Delta(\lambda) = \lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda$ and $\lambda = 0$ is an eigenvalue. In this case, the matrix is singular and its inverse does not exist.

3. a) $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ $K_0 = 0$, $K = 3$

$$x(k_0) = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \qquad x(K) = \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$$

Since this is an LTI system, we can use the result from Chen exercise 3.11 (Last homework, problem 4) to write

$$x(3) = \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} = bu(2) + Abu(1) + A^2bu(0) + A^3x(0)$$

hence

$$\begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} u(2) + \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} u(1) + \begin{bmatrix} 4 \\ 4 \\ -4 \end{bmatrix} u(0) + \begin{bmatrix} 6 & 7 & 5 \\ 5 & 6 & 3 \\ -3 & -5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

rearrange ...

$$\underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \\ -1 & -2 & -4 \end{bmatrix}}_{\triangleq Q} \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}_{W}$$

clearly, the columns of Q are linearly dependent. The range of Q may be expressed by the basis

$$V_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\} \quad \text{(one dimensional)}$$

observe that:
$W \in \text{range}(Q)$ hence $\exists$ an input sequence, e.g. $u(2) = 1$, $u(1) = 0$, $u(0) = 0$ that satisfies the requirements. The input sequence is not unique though. Another valid input sequence is $u(2) = 0$, $u(1) = \frac{1}{2}$, $u(0) = 0$.

b) A same as part (a), $b = [1\ 1\ 1]^T$.
K_0 and K same as part a

$$x(K_0) = \begin{bmatrix} -1 \\ 2 \\ -4 \end{bmatrix} \qquad x(K) = \begin{bmatrix} 2 \\ -6 \\ 1 \end{bmatrix}$$

do same steps as part a, ...

$$\underbrace{\begin{bmatrix} 1 & 4 & 8 \\ 1 & 2 & 6 \\ 1 & 0 & -2 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} 14 \\ -1 \\ 8 \end{bmatrix}}_{w}$$

In this case, Gaussian elimination shows that Q is invertible, hence $u = Q^{-1}w$ and $u = [0\ \ 11\tfrac{1}{2}\ \ -4]^T$ (u is unique).

c) $A(k) = \begin{bmatrix} \cos(\pi k) & \frac{1}{2} \\ 0 & \sin(\frac{\pi k}{2}) \end{bmatrix}$ $b(k) = \begin{bmatrix} k \\ 1 \end{bmatrix}$

$k_0 = 2$, $k = 4$, $x(k_0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x(k) = \begin{bmatrix} 41 \\ -19 \end{bmatrix}$

Recall that for a time varying linear discrete time system,

$$x(k) = \Phi(k, k_0)x(k_0) + \sum_{l=k_0}^{k-1} \Phi(k, l+1) B(l) u(l)$$

where $\Phi(k,j) = \begin{cases} I_2 & \text{if } k=j \\ A(k-1)A(k-2)\cdots A(j) & \text{if } k>j \end{cases}$

hence, in our case:

$$x(4) = \begin{bmatrix} 41 \\ -19 \end{bmatrix} = \Phi(4,2)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \Phi(4,3)B(2)u(2) + \Phi(4,4)B(3)u(3)$$

$$\Phi(4,2) = A(3)A(2) = \begin{bmatrix} \cos(\pi 3) & \frac{1}{2} \\ 0 & \sin(\frac{\pi 3}{2}) \end{bmatrix}\begin{bmatrix} \cos(\pi 2) & \frac{1}{2} \\ 0 & \sin(\frac{\pi 2}{2}) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & \frac{1}{2} \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

$$\Phi(4,3) = A(3) = \begin{bmatrix} -1 & \frac{1}{2} \\ 0 & -1 \end{bmatrix}$$

$$\Phi(4,4) = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$B(2) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $B(3) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

now, plug it all in...

$$\begin{bmatrix} 41 \\ -19 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 & \frac{1}{2} \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}u(2) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix}u(3)$$

rearrange...

$$\underbrace{\begin{bmatrix} -3/2 & 3 \\ -1 & 1 \end{bmatrix}}_{Q}\begin{bmatrix} u(2) \\ u(3) \end{bmatrix} = \begin{bmatrix} 41 + \frac{3}{2} \\ -19 \end{bmatrix}$$

Q is invertible, hence $\exists$ a unique solution to the problem:

$u(2) = 66\frac{1}{3}$, $u(3) = 47\frac{1}{3}$

4. $\dot{x}(t) = (t-1)(x(t)+1)\ ;\ x(t_0)=0$

$$= (t-1)x(t) + (t-1) \quad ;\ x(t_0)=0$$

just use the result from lecture for $\dot{x}(t) = a(t)x(t) + b(t)u(t)$

where $a(t) = t-1$ and $b(t)u(t) = t-1$

Then : $$x(t) = \phi(t,t_0)\,x(t_0) + \int_{t_0}^{t} \phi(t,\tau)\,b(\tau)u(\tau)\,d\tau$$

where $$\phi(t,s) = \exp\left\{\int_s^t a(\tau)\,d\tau\right\}$$

Since $x(t_0)=0$, we can ignore the first term, hence

$$x(t) = \int_{t_0}^{t} \phi(t,\tau)\,b(\tau)u(\tau)\,d\tau = \int_{t_0}^{t} \exp\Big\{\underbrace{\int_\tau^t (x-1)\,dx}_{\text{first do this integral}}\Big\}(\tau-1)\,d\tau$$

$$\int_\tau^t (x-1)\,dx = \left(\frac{x^2}{2} - x\right)\Bigg|_\tau^t = \frac{t^2}{2} - t - \frac{\tau^2}{2} + \tau$$

hence

$$x(t) = \exp\left\{\frac{t^2}{2} - t\right\}\int_{t_0}^{t} \exp\left(\tau - \frac{\tau^2}{2}\right)(\tau-1)\,d\tau$$

$$= -\exp\left\{\frac{t^2}{2} - t\right\}\left[\exp\left\{\tau - \frac{\tau^2}{2}\right\}\right]_{\tau=t_0}^{\tau=t}$$

$$= -\exp\left\{\frac{t^2}{2} - t\right\}\left[\exp\left\{t - \frac{t^2}{2}\right\} - \exp\left\{t_0 - \frac{t_0^2}{2}\right\}\right]$$

$\exp(x)\cdot\exp(-x) = 1$

hence

$$x(t) = \exp\left\{\frac{t^2 - t_0^2}{2} - (t-t_0)\right\} - 1$$

check :

$x(t_0) = \exp(0) - 1 = 0$ ✓

$$\frac{d}{dt}x(t) = \exp\left\{t_0 - \frac{t_0^2}{2}\right\}\frac{d}{dt}\left[\exp\left\{\frac{t^2}{2} - t\right\}\right]$$

$$= \exp\left\{t_0 - \frac{t_0^2}{2}\right\}(t-1)\exp\left\{\frac{t^2}{2} - t\right\}$$

$$= (t-1)\exp\left\{\frac{t^2 - t_0^2}{2} - (t-t_0)\right\}$$

$$= (t-1)\left[\exp\left\{\frac{t^2 - t_0^2}{2} - (t-t_0)\right\} - 1\right] + t - 1 = (t-1)x(t) + (t-1)$$ ✓

5. $\dot{x}(t) = Ax(t)$

$x(t) = e^{At} x(0)$

experiment #1 : $x(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3e^{-2t} \\ e^{-2t} \end{bmatrix} = e^{At} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (1)

experiment #2 : $x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} e^{-4t} \\ 2e^{-4t} \end{bmatrix} = e^{At} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (2)

let $e^{At} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then we can rewrite (1) and (2) as

$$\begin{bmatrix} 3e^{-2t} & e^{-4t} \\ e^{-2t} & 2e^{-4t} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \underbrace{\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}}_{P}, \quad P^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$$

$$\frac{1}{5}\begin{bmatrix} 3e^{-2t} & e^{-4t} \\ e^{-2t} & 2e^{-4t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

hence $e^{At} = \frac{1}{5}\begin{bmatrix} 6e^{-2t} - e^{-4t} & -3e^{-2t} + 3e^{-4t} \\ 2e^{-2t} - 2e^{-4t} & -e^{-2t} + 6e^{-4t} \end{bmatrix} = \Phi(t,0)$

check that $\Phi(0,0) = I_2$, $\Phi(0,0) = \frac{1}{5}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = I_2$ ✓

Now, $\frac{d}{dt}\Phi(t,\tau) = A\,\Phi(t,\tau)$ hence we just need to compute $\frac{d}{dt}\Phi(t,\tau)$ to find A.

$$\frac{d}{dt}\Phi(t,\tau) = \frac{1}{5}\begin{bmatrix} -12e^{-2t} + 4e^{-4t} & 6e^{-2t} - 12e^{+4t} \\ -4e^{-2t} + 8e^{-4t} & 2e^{-2t} - 24e^{-4t} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Phi(t,t)$$

or

$$\begin{bmatrix} -12e^{-2t} + 4e^{-4t} & 6e^{-2t} - 12e^{-4t} \\ -4e^{-2t} + 8e^{-4t} & 2e^{-2t} - 24e^{-4t} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 6e^{-2t} - e^{-4t} & 3e^{-2t} + 3e^{-4t} \\ 2e^{-2t} - 2e^{-4t} & -e^{-2t} + 6e^{-4t} \end{bmatrix}$$

set exponential coefficients equal ...

$$\left.\begin{array}{l} 6a_{11} + 2a_{12} = -12 \\ -a_{11} - 2a_{12} = 4 \end{array}\right\} \Rightarrow a_{11} = -\frac{8}{5} \quad a_{22} = -\frac{6}{5}$$

$$\left.\begin{array}{l} 6a_{21} + 2a_{22} = -4 \\ -a_{21} - 2a_{22} = 8 \end{array}\right\} \Rightarrow a_{21} = \frac{4}{5} \quad a_{22} = -\frac{22}{5}$$

hence $A = \begin{bmatrix} -8/5 & -6/5 \\ \frac{4}{5} & -\frac{22}{5} \end{bmatrix}$

Solution to problem 6:

We want $x(t) = x(t_0)$ for all t. This implies that

$\dot{x}(t) \equiv 0$ for all t.

Then $0 = A(t)x(t) + B(t)u(t)$. Since $x(t) \equiv x(t_0)$ $\forall t$, we can rearrange this equation to write

$$B(t)u(t) = -A(t)x(t_0)$$ [we always assume we know $A(t)$, $B(t)$]

Since we are given $x(t_0)$, a solution to this equation exists for $u(t)$ if and only if

$$A(t)x(t_0) \in \text{range}(B(t)) \text{ for all } t \in \mathbb{R}. \quad (1)$$

The solution, if it exists, is unique iff

$$\dim(\text{nullspace}(B(t))) = 0 \text{ for all } t \in \mathbb{R}. \quad (2)$$

Now look at $\dot{x}(t) = x(t) + e^{-t}u(t)$, rewrite...

$$0 = x(t_0) + e^{-t}u(t)$$

hence $e^{-t}u(t) = -x(t_0)$ ← scalar equation, $e^{-t} \neq 0$ for all $t \in \mathbb{R}$ (eqs (1) and (2) satisfied)

$u(t) = -e^{t}x(t_0)$. $\Rightarrow$ $\dot{x}(t) = x(t) - x(t_0)$

check: here $A(t) = 1$
$B(t)u(t) = -x(t_0)$

then $$x(t) = e^{t-t_0}x(t_0) - \int_{t_0}^{t} e^{(t-\tau)}x(t_0)\,d\tau$$

$$= e^{t-t_0}x(t_0) - e^{t}x(t_0)\int_{t_0}^{t} e^{-\tau}\,d\tau$$

$$= x(t_0)\left[e^{t-t_0} - e^{t}\left(e^{-t_0} - e^{-t}\right)\right]$$

$$= x(t_0)\left[e^{t-t_0} - e^{t-t_0} + 1\right]$$

$$= x(t_0) \text{ for all } t \in \mathbb{R}. \checkmark$$

7. a) use (STM), let

$$\Phi(t,t_0) = \begin{bmatrix} \lambda_{11}(t,t_0) & \lambda_{12}(t,t_0) \\ \lambda_{21}(t,t_0) & \lambda_{22}(t,t_0) \end{bmatrix} \qquad (*)$$

Then by (STM),

$$\frac{d}{dt}\Phi(t,t_0) = \begin{bmatrix} \frac{d}{dt}\lambda_{11} & \frac{d}{dt}\lambda_{12} \\ \frac{d}{dt}\lambda_{21} & \frac{d}{dt}\lambda_{22} \end{bmatrix} = \begin{bmatrix} 3t & 0 \\ t & 0 \end{bmatrix}\begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}$$

hence $\frac{d}{dt}\lambda_{11}(t,t_0) = 3t\,\lambda_{11}(t,t_0)$ and $\lambda_{11}(t,t_0) = 1$ since

$\Phi(t,t_0) = I$ (by STM)

the solution to this differential equation is $\boxed{\lambda_{11}(t,t_0) = \exp\left(\frac{3t^2}{2} - \frac{3t_0^2}{2}\right)}$

(check: $\frac{d}{dt}\left(\exp\left(\frac{3t^2}{2} - \frac{3t_1^2}{2}\right)\right) = 3t\exp(\sim)$ ✓

$\exp\left(\frac{3t^2}{2} - \frac{3t_0}{2}\right) = 1$ when $t = t_0$ ✓)

now

$\frac{d}{dt}\lambda_{12}(t,t_0) = 3t\,\lambda_{12}$ → looks the same as last case except $\lambda_{12}(t_0,t_0) = 0$.

the solution is then $\boxed{\lambda_{12}(t,t_0) = 0}$

(check, when $t = t_0$ then $\lambda_{12}(t,t_0) = 0$ ✓

$\frac{d}{dt}\lambda_{12}(t,t_0) = 3t \cdot 0$ ✓)

now

$\frac{d}{dt}\lambda_{21}(t,t_0) = t\,\lambda_{11}(t,t_0)$ with $\lambda_{21}(t_0,t_0) = 0$.

We already know $\lambda_{11}(t,t_0)$ from above, hence we need to find

$\frac{d}{dt}\lambda_{21}(t,t_0) = t\exp\left(\frac{3t^2}{2} - \frac{3t_0^2}{2}\right)$

solution: $\boxed{\lambda_{21}(t,t_0) = \frac{1}{3}\exp\left(\frac{3}{2}t^2 - \frac{3}{2}t_0^2\right) - \frac{1}{3}}$

check $\lambda_{21}(t_0,t_0) = 0$ ✓

$\frac{d}{dt}\lambda_{21}(t,t_0) = 3t \cdot \frac{1}{3}\exp\left(\frac{3}{2}t^2 - \frac{3}{2}t_0^2\right)$

$= t\,\lambda_{11}(t,t_0)$ ✓

finally

$\frac{d}{dt}\lambda_{22}(t,t_0) = t\,\lambda_{12}(t,t_0)$ with $\lambda_{22}(t_0,t_0) = 1$.

from results above, $\frac{d}{dt}\lambda_{22}(t,t_0) = 0$, but $\lambda_{22}(t_0,t_0) = 1$ implies that

$\boxed{\lambda_{22}(t,t_0) = 1}$ (checks trivially) hence

$$\Phi(t,t_0) = \begin{bmatrix} \exp\left(\frac{3t^2}{2} - \frac{3t_0^2}{2}\right) & 0 \\ \frac{1}{3}\exp\left(\frac{3t^2}{2} - \frac{3t_0^2}{2}\right) - \frac{1}{3} & 1 \end{bmatrix}$$

b) In this case $t_0 = 3$, $t = 2$ (going backwards in time!)

$x(t) = \Phi(t,t_0)\, x(t_0)$ ← this is defined for all $t, t_0 \in \mathbb{R}$ (unlike the discrete time case).

↑ we have this from part a.

hence, just plug in numbers...

$$x(2) = \Phi(2,3)\, x(3)$$

$$= \begin{bmatrix} \exp\left(6 - \frac{27}{2}\right) & 0 \\ \frac{1}{3}\exp\left(6 - \frac{27}{2}\right) - \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 5.531 \times 10^{-4} \\ 0.6669 \end{bmatrix}$$

Solution to problem 8 :

a) $A = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

characteristic polynomial: $\det(\lambda I_3 - A) = \begin{vmatrix} \lambda-1 & -4 & -10 \\ 0 & \lambda-2 & 0 \\ 0 & 0 & \lambda-2 \end{vmatrix} = (\lambda-1)(\lambda-2)^2$

hence the e-values are $\lambda_1 = 1$ and $\lambda_2 = 2$.
The algebraic multiplicities are $r_1 = 1$ and $r_2 = 2$.
Now find bases for eigenspaces...

$E(\lambda_1)$: $A - \lambda_1 I_3 = \begin{bmatrix} 0 & 4 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, by inspection $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

is in the nullspace of $A - \lambda_1 I_3$. Since the geometric multiplicity of λ_1 is upper bounded by $r_1 = 1$, we don't need to search for any more basis vectors. A basis for $E(\lambda_1)$ is then $\{v_1\}$.

$E(\lambda_2)$: $A - \lambda_2 I_3 = \begin{bmatrix} -1 & 4 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, by inspection, we can quickly find two linearly independent vectors in the nullspace of $A - \lambda_2 I_3$.

$v_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} 10 \\ 0 \\ 1 \end{bmatrix}$ $\longrightarrow$ clearly v_2 and v_3 are linearly independent and in nullspace$(A - \lambda_2 I_3)$

Since the geometric multiplicity of λ_2 is upper bounded by $r_2 = 2$, we don't need to search for any more basis vectors, hence a basis for $E(\lambda_2)$ is then $\{v_2, v_3\}$.

The geometric multiplicities are then $m_1 = 1$ and $m_2 = 2$.
This matrix is diagonalizable since $r_j = m_j$ for all $j \in \{1,2\}$.
In fact,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = V^{-1} A V \quad \text{where } V = [v_1 \ v_2 \ v_3]$$

b) $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}$, char polynomial $= \det(\lambda I_4 - A)$

$$= \begin{vmatrix} \lambda & -1 & 0 & 0 \\ -2 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ 1 & 0 & 0 & \lambda \end{vmatrix}$$

$$= \lambda \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda \end{vmatrix} + \begin{vmatrix} -2 & -1 & 0 \\ 0 & \lambda & -1 \\ 1 & 0 & \lambda \end{vmatrix}$$

cont...

$$= \lambda^4 + (-2\lambda^2 + 1) = \lambda^4 - 2\lambda^2 + 1 = (\lambda^2 - 1)^2 = [(\lambda-1)(\lambda+1)]^2$$
$$= (\lambda-1)^2(\lambda+1)^2$$

hence the e-values are $\lambda_1 = -1$, $\lambda_2 = 1$.
The algebraic multiplicities are $m_1 = 2$ and $m_2 = 2$.

Now find bases for eigenspaces:

$\underline{E(\lambda_1)}$: $A - \lambda_1 I_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \tilde{A}$

best approach is to find echelon form of $\tilde{A}$...
do Gaussian elimination...

$$\tilde{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \tilde{A}_e$$

$\tilde{A}_e$ has only one non-pivot column hence $\dim(\text{nullspace}(\tilde{A})) = 1$.
By inspection, an e-vector of A is then

$$V_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad \text{(check it!)}$$

There are no other linearly independent e-vectors corresponding to λ_1. A basis for $E(\lambda_1)$ is then $\{V_1\}$.

$\underline{E(\lambda_2)}$: $A - \lambda_2 I_4 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & -1 \end{bmatrix} = \tilde{A}$

just like before, do GE...

$$\tilde{A}_e = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

only one non pivot column.
hence $\dim(\text{nullspace}(\tilde{A})) = 1$.

An e-vector of A is then $V_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$

There are no other linearly indep. e-vectors corresponding to λ_2. Hence a basis for $E(\lambda_2)$ is then $\{V_2\}$.

The geometric multiplicities are then $m_1 = 1$ and $m_2 = 1$.
This matrix is not diagonalizable since $\exists$ at least one j such that $r_j \neq m_j \Rightarrow$ nondiagonalizable matrix.

c) $A = I_n$. Characteristic polynomial is simply $(\lambda-1)^n$, hence there is only one unique e-value: $\lambda_1 = 1$.
The algebraic multiplicity is then $r_1 = n$.

<u>$E(\lambda_1)$:</u> $A - \lambda_1 I_n = \underbrace{\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}}_{\substack{n\times n \text{ matrix} \\ \text{full of} \\ \text{zeros.}}} = \tilde{A}$

Nullspace $(\tilde{A})$ is n-dimensional and we can pick any n linearly independent vectors as a basis for $\mathbb{R}^n$.

For instance, a basis for $E(\lambda_1)$ is $\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{n \text{ vectors in } \mathbb{R}^n} \right\}$

The geometric multiplicity of λ_1 is then $m_1 = n$.

A is diagonalizable since $r_1 = m_1$. (A is already diagonal!)

No. 5505 Engineer's Computation Pad

Solution to problem 9 :

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} x(t)$$

$$y(t) = [1 \quad 1 \quad 1]\, x(t)$$

Since $x(t) = e^{At}x(0)$ in this case, we will need to compute e^{At} ... we will use e-value / e-vector method.

characteristic polynomial $= (\lambda+1)^2(\lambda-1)$ ← repeated roots! might not be diagonalizable.

let $\lambda_1 = -1$, $\lambda_2 = 1$

$$E(\lambda_1) = \text{nullspace}(A - \lambda_1 I) = \text{nullspace}\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

clearly $(A - \lambda_1 I)$ has a 2-dimensional nullspace that can be described by the basis $\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{v_2} \right\}$

hence $\dim(E(\lambda_1)) = 2 = \underbrace{m_1}_{\text{geometric multiplicity}} = \underbrace{r_1}_{\text{algebraic multiplicity}}$

Easy to verify $\dim(E(\lambda_2)) = 1 = m_2 = r_2$, hence A is diagonalizable.

A basis for $E(\lambda_2) = \text{nullspace}(A - \lambda_2 I) = \text{nullspace}\left(\begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}\right)$

$V_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \in E[\lambda_2]$ (check it!)

Let $V = [v_1 \quad v_2 \quad v_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$, then $AV = V\Lambda$

where $\Lambda = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A = V\Lambda V^{-1}$

since A is diagonalizable, $e^{At} = Ve^{\Lambda t}V^{-1}$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & -\frac{1}{2}e^{-t} \\ 0 & e^{-t} & -e^{-t} \\ 0 & 0 & \frac{1}{2}e^{t} \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 & -\frac{1}{2}e^{-t} + \frac{1}{2}e^{t} \\ 0 & e^{-t} & -e^{-t} + e^{t} \\ 0 & 0 & e^{t} \end{bmatrix} = e^{At}$$

hence, $y(t) = C e^{At} x(0) = [1 \ 1 \ 1] \begin{bmatrix} e^{-t} & 0 & -\frac{1}{2}e^{-t}+\frac{1}{2}e^{t} \\ 0 & e^{-t} & -e^{-t}+e^{t} \\ 0 & 0 & e^{t} \end{bmatrix} x(0)$

$$y(t) = \begin{bmatrix} e^{-t} & e^{-t} & -\frac{3}{2}e^{-t}+\frac{5}{2}e^{t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$$

to get $y(t) = 3e^{-t}$ we must have

$x_1(0) + x_2(0) - \frac{3}{2}x_3(0) = 3$

and $x_3(0) = 0$

hence $\left.\begin{array}{r} x_1(0)+x_2(0) = 3 \\ x_3(0) = 0 \end{array}\right\}$ satisfies the problem

A solution <u>exists</u>, e.g. $x(0) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ but it is <u>not unique</u>.

Another solution is $x(0) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Note that the set of all possible $x(0)$ satisfying the problem is <u>not</u> a <u>subspace</u> of $\mathbb{R}^3$.

b) $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t)$

$y(t) = [1 \ 1] x(t)$

A is not diagonalizable but we can compute e^{At} since

$\frac{d}{dt}\Phi(t,0) = A(t)\,\Phi(t,0)$ and $\Phi(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

here we have $\Phi(t,0) = e^{At}$ since our system is time invariant

$$\begin{bmatrix} \dot{\phi}_{11}(t,0) & \dot{\phi}_{12}(t,0) \\ \dot{\phi}_{21}(t,0) & \dot{\phi}_{22}(t,0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_{11}(t,0) & \phi_{12}(t,0) \\ \phi_{21}(t,0) & \phi_{22}(t,0) \end{bmatrix}$$

$\dot{\phi}_{11}(t,0) = 0$ but $\phi_{11}(0,0) = 1 \implies \phi_{11}(t,0) = 1$

$\dot{\phi}_{21}(t,0) = 0$ and $\phi_{21}(0,0) = 0 \implies \phi_{21}(t,0) = 0$

$\dot{\phi}_{22}(t,0) = 0$ but $\phi_{22}(0,0) = 1 \implies \phi_{22}(t,0) = 1$

$\dot{\phi}_{12}(t,0) = \phi_{22}(t,0)$ and $\phi_{12}(0,0) = 0 \implies \phi_{12}(t,0) = t$

hence $e^{At} = \Phi(t,0) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

now $y(t) = C e^{At} x(0) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$

$$= \begin{bmatrix} 1 & t+1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$= x_1(0) + (t+1)\, x_2(0)$$

we want $y(t) = t/2$, hence $\left.\begin{array}{l} x_1(0) + x_2(0) = 0 \\ x_2(0) = 1/2 \end{array}\right\} \Rightarrow x_1(0) = -1/2$

$x(0) = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$ is the unique solution to this problem.