Solution to Problem 1 : (chen 5.11)

$$\dot{x} = \underbrace{\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A} x$$

Test for "marginal" stability: Look at e-values of A

characteristic polynomial

$$\det(\lambda I_3 - A) = \det\begin{bmatrix} \lambda+1 & 0 & -1 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} = (\lambda+1)\lambda^2$$

we have one e-value at $\lambda_1 = -1$ and two at $\lambda_2 = 0$

$r_1 = 1$, $r_2 = 2$

The e-value at $\lambda_1 = -1$ is no problem but the pair of e-values at $\lambda_2 = 0$ is a problem - we need to check that $m_2 = r_2$ (the geometric multiplicity equals the algebraic multiplicity for the λ_2 e-value).

$$E(\lambda_2) = \text{nullspace}(A - \lambda_2 I_3) = \text{nullspace}(A)$$

A is already in echelon form hence a basis for the nullspace follows naturally as

$$\mathcal{B}_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Since we have a 1 dimesional basis $m_2 = 1 \neq r_2$.

$\Rightarrow$ A is <u>not</u> "marginally" stable.

Note that A is <u>not</u> asymptotically stable since A is not marginally stable.

Solution to Problem 2 : Chen 5.12

$$x(k+1) = \begin{bmatrix} 0.9 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(k)$$

Test for "marginal" stability: Again look at e-values of A

we see that $\lambda_1 = 0.9$, $\lambda_2 = 1$ $\quad \underbrace{r_1 = 1, \; r_2 = 2}_{\text{algebraic multiplicities}}$

Note that λ_1 is no problem but the repeated e-value $\lambda_2 = 1$ might be. Need to check geometric multiplicity of λ_2.

$$E(\lambda_2) = \text{null}\left(A - \lambda_2 I_3\right) = \text{null}\left(\begin{bmatrix} -0.1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

A basis for $E(\lambda_2)$ is $\mathcal{B}_2 = \left\{ \begin{bmatrix} 10 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\dim(E(\lambda_2)) = m_2 = 2$

and since $m_2 = r_2$, this system is "marginally" stable.

This system is <u>not</u> asymptotically stable due to the e-value $\lambda_2 = 1$.

<u>Solution to Problem 3</u> : Chen 5.14

$$A = \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix}$$

We need to use the Lyapunov stability theorem to show that the e-values of A have negative real parts.

We will use Lyapunov Lemma II which says that if

$$A^T P + PA = -Q$$

has a unique positive definite solution for P for any particular positive definite Q, then A's e-values have negative real parts.

Let $Q = I_2$

Then we must find $p_{11}, p_{12}, p_{21}, p_{22}$ satisfying

$$\begin{bmatrix} 0 & -0.5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

To solve this, first simplify the LHS

$$\begin{bmatrix} -\frac{1}{2} p_{21} & -\frac{1}{2} p_{22} \\ p_{11} - p_{21} & p_{12} - p_{22} \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} p_{12} & p_{11} - p_{12} \\ -\frac{1}{2} p_{22} & p_{21} - p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

hence, equating (i,j)th entries, we have 4 equations:

a) $-\frac{1}{2}P_{21} - \frac{1}{2}P_{12} = -1$

b) $-\frac{1}{2}P_{22} + P_{11} - P_{12} = 0$

c) $P_{11} - P_{21} - \frac{1}{2}P_{22} = 0$

d) $P_{12} + P_{21} - 2P_{22} = -1$

since solution for P must be positive definite, we require symmetry hence $P_{21} = P_{12}$. We can solve a) immediately then

$$-\frac{1}{2}P_{21} - \frac{1}{2}P_{12} = -1 \Rightarrow P_{12} = P_{21} = 1$$

Then equation d) $\Rightarrow -2P_{22} = -3$, $P_{22} = \frac{3}{2}$

then equation b) $\Rightarrow P_{11} = 1 + \frac{1}{2}\cdot\frac{3}{2} = \frac{7}{4}$

Note that equation c) is also satisfied now.

hence $P = \begin{bmatrix} 7/4 & 1 \\ 1 & 3/2 \end{bmatrix}$ which satisfies the symmetry property but is it positive definite?

brute force...

$$\det(\lambda I - P) = \det\left(\begin{bmatrix} \lambda - 7/4 & -1 \\ -1 & \lambda - 3/2 \end{bmatrix}\right) = (\lambda - \tfrac{7}{4})(\lambda - \tfrac{3}{2}) - 1$$

$$= \lambda^2 - \frac{13}{4}\lambda + \frac{21}{8} - 1 = \lambda^2 - \frac{13}{4}\lambda + \frac{13}{8}$$

roots of this quadratic equation are

$$\frac{\frac{13}{4} \pm \sqrt{\left(\frac{13}{4}\right)^2 - \frac{13}{2}}}{2} = \frac{\frac{13}{4} \pm \sqrt{\frac{169}{16} - \frac{104}{16}}}{2}$$

$$= \frac{\frac{13}{4} \pm \sqrt{\frac{65}{16}}}{2} = \frac{13}{8} \pm \frac{\sqrt{65}}{8}$$

Since $\sqrt{65} < 13$ then both roots are positive

and P is positive definite. Unique positive definite P implies that A is then Hurwitz.

(Note that there are easier ways to show P is positive defn...)

Solution to Problem 4.

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

use Lyapunov Stability theorem

$$P - A^T P A = Q \quad , \text{let } Q = I_2$$

Then

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

just pull out the $\frac{1}{2}$ terms

simplify...

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} P_{11} - P_{12} & P_{11} + P_{12} \\ P_{21} - P_{22} & P_{21} + P_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} - \frac{1}{4}\begin{bmatrix} P_{11} - P_{12} - P_{21} + P_{22} & P_{11} + P_{12} - P_{21} - P_{22} \\ P_{11} - P_{12} + P_{21} - P_{22} & P_{11} + P_{12} + P_{21} + P_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

equate (i,j)th terms:

$$\frac{3}{4}P_{11} + \frac{1}{4}P_{12} + \frac{1}{4}P_{21} - \frac{1}{4}P_{22} = 1$$

$$-\frac{1}{4}P_{11} + \frac{3}{4}P_{12} + \frac{1}{4}P_{21} + \frac{1}{4}P_{22} = 0$$

$$-\frac{1}{4}P_{11} + \frac{1}{4}P_{12} + \frac{3}{4}P_{21} + \frac{1}{4}P_{22} = 0$$

$$-\frac{1}{4}P_{11} - \frac{1}{4}P_{12} - \frac{1}{4}P_{21} + \frac{3}{4}P_{22} = 1$$

Use Matlab to solve...

$$\begin{aligned} P_{11} &= \tfrac{1}{2} \\ P_{12} = P_{21} &= 0 \\ P_{22} &= \tfrac{1}{2} \end{aligned} \quad \Rightarrow \quad P = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

P clearly positive definite and unique, hence A is Hurwitz. (as expected)

Now look at $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, use prior analysis to write:

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} - \begin{bmatrix} p_{11} - p_{12} - p_{21} + p_{22} & p_{11} + p_{12} - p_{21} - p_{22} \\ p_{11} - p_{12} + p_{21} - p_{22} & p_{11} + p_{12} + p_{21} + p_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

equate (i,j)th terms:

$$\begin{aligned} p_{12} + p_{21} - p_{22} &= 1 \\ -p_{11} + p_{21} + p_{22} &= 0 \\ -p_{11} + p_{12} + p_{22} &= 0 \\ -p_{11} - p_{12} - p_{21} &= 1 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}}_{X} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{21} \\ p_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

X is invertible, so unique soln exists,

$$\left.\begin{aligned} p_{11} &= -1 \\ p_{12} &= 0 \\ p_{21} &= 0 \\ p_{22} &= -1 \end{aligned}\right\} \Rightarrow P = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

P not positive definite hence A not Hurwitz.

(as expected)

Solution to problem 5:

$$x(k+1) = \underbrace{\begin{bmatrix} \cos\theta & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\theta \end{bmatrix}}_{A} x(k)$$

Lets look at asymptotic stability first. Find the e-values of A.

$$\det(\lambda I_2 - A) = (\lambda - \cos\theta)^2 + \sin^2\frac{\theta}{2}$$

$$= \lambda^2 - 2\lambda\cos\theta + \cos^2\theta + \sin^2\frac{\theta}{2}$$

quadratic equation, roots:

$$\text{roots} = \frac{2\cos\theta \pm \sqrt{\cancel{4\cos^2\theta} - \cancel{4\cos^2\theta} - 4\sin^2\frac{\theta}{2}}}{2}$$

$$= \frac{2\cos\theta \pm j\,2\sin\frac{\theta}{2}}{2} = \cos\theta \pm j\sin\frac{\theta}{2}$$

what values of θ cause both roots $\cos\theta + j\sin\frac{\theta}{2} = \lambda_1$, $\cos\theta - j\sin\frac{\theta}{2} = \lambda_2$ to have magnitude less than one?

Since both roots have same magnitude, we just need to look at one of them.

$$\left|\cos\theta + j\sin\frac{\theta}{2}\right| = \sqrt{\cos^2\theta + \sin^2\frac{\theta}{2}} < 1$$

square both sides

$$\cos^2\theta + \sin^2\frac{\theta}{2} < 1$$

trig identity $\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$ hence

$$\cos^2\theta + \frac{1}{2}(1-\cos\theta) < 1 \iff \cos^2\theta - \frac{1}{2}\cos\theta + \frac{1}{2} < 1$$

$$\cos^2\theta - \frac{1}{2}\cos\theta < \frac{1}{2} \iff \cos^2\theta - \frac{1}{2}\cos\theta + \frac{1}{16} < \frac{1}{2} + \frac{1}{16}$$

$$\left(\cos\theta - \frac{1}{4}\right)^2 < \frac{1}{2} + \frac{1}{16} \iff \left(\cos\theta - \frac{1}{4}\right)^2 < \frac{9}{16}$$

$$-\sqrt{\frac{9}{16}} < \cos\theta - \frac{1}{4} < \sqrt{\frac{9}{16}}$$

$$-\frac{1}{2} < \cos\theta < 1$$

hence, since $\cos\theta = 1$ only when $\theta = 0$ (in our range of $\theta \in [0, 2\pi)$) then we have asymptotic stability when

$\theta \in (0, \cos^{-1}(-\frac{1}{2}))$

and by symmetry

$\theta \in (2\pi - \cos^{-1}(-\frac{1}{2}), 2\pi)$.

"Marginal" stability is satisfied at all of these points also but we need to check the points

$$\theta = \{0, \cos^{-1}(-\tfrac{1}{2}), 2\pi - \cos^{-1}(-\tfrac{1}{2})\}$$

When $\theta = 0$, $A = I_2$, repeated e-values but diagonalizable, hence "marginally" stable.

When $\theta = \cos^{-1}(-\frac{1}{2})$, $A = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

in this case A has 2 distinct evalues $\lambda_1 = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$
$\lambda_2 = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$

both have magnitude equal to one but $m_1 = r_1$ and $m_2 = r_2$ hence "marginally stable".

When $\theta = 2\pi - \cos^{-1}(-\frac{1}{2})$, $A = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ same as prior case.

Hence system is "marginally" stable for $\theta \in [0, \cos^{-1}(-\frac{1}{2})]$
and $\theta \in [2\pi - \cos^{-1}(-\frac{1}{2}), 2\pi)$.

end of solution