1. Assume $\bar{x}$ unobservable. Then $Ce^{tA}\bar{x}=0 \quad \forall t \geq 0$. Since e^{tA} is a continuously differentiable function of t, we can take derivatives of $Ce^{tA}\bar{x}$ and we know that all of them are equal to zero $\forall\ t \geq 0$. Evaluating at $t=0$ we get

$$\begin{aligned} C\bar{x} &= 0 && \text{(no derivative , } t=0) \\ CA\bar{x} &= 0 && \text{(first derivative , } t=0) \\ CA^2\bar{x} &= 0 && \text{(2nd derivative , } t=0) \\ &\ \vdots && \quad \vdots \\ CA^{n-1}\bar{x} &= 0 && \text{(n-1'th derivative , } t=0) \end{aligned}$$

$$\Rightarrow \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \bar{x} = 0 \iff Q_o\bar{x} = 0 \quad \text{hence } \bar{x} \in null(Q_o)$$

Conversely, assume $\bar{x} \in null(Q_o)$. Then $CA^k\bar{x} = 0$ for all $k = 0, \dots, n-1$. Now express

$$e^{tA} = \sum_{k=0}^{n-1} p_k(t) A^k$$

by the C-H Thm.

$$Ce^{tA}\bar{x} = C\sum_{k=0}^{n-1} p_k(t) \underbrace{A^k\bar{x}}_{\substack{=0 \\ \text{for all } k=0,\dots,n-1}} = 0 \implies \bar{x} \text{ unobservable.}$$

2. If system $\{A, B, C, D\}$ not observable then $rank(Q_o) = r < n$ and $\dim(null(Q_o)) = n-r > 0$.
Let $\{w_1, w_2, \dots, w_{n-r}\}$ be a basis for $null(Q_o)$ and let $\{v_1, v_2, \dots, v_r\}$ be a basis for the rest of $\mathbb{R}^n$ such that $\{v_1, v_2, \dots, v_r, w_1, \dots, w_{n-r}\}$ is a basis for $\mathbb{R}^n$.

Let $P = [v_1\ v_2\ \dots\ v_r\ \ w_1\ \dots\ w_{n-r}]$. P is invertible.

since $w_k \in null(Q_o)$ $k=1,\dots,n-r$

$$\text{Then } Q_oP = \begin{bmatrix} CP \\ CAP \\ CA^2P \\ \vdots \\ CA^{n-1}P \end{bmatrix} = \begin{bmatrix} \bar{C} & 0 \\ ? & 0 \\ & \vdots \\ & 0 \end{bmatrix} \begin{matrix} \}P \\ \\ \\ \\ \end{matrix} \to \text{set } [\underbrace{\bar{C}}_{r} \ \ \underbrace{0}_{n-r}] = CP$$

with the column blocks of Q_oP of widths r and $n-r$.

these zeros are due to the fact that the last $n-r$ columns of AP are linear combinations of the last $n-r$ columns of P.

$$\text{Then } AP = \Big[\ \cdots \ \ \underbrace{\ \cdots\ }_{n-r}\Big] = P\begin{bmatrix} \bar{A} & 0 \\ & \vdots \\ & 0 \\ ? & ? \end{bmatrix} \begin{matrix} \}r \\ \\ \}n-r \end{matrix}$$

with the column blocks of widths r and $n-r$.

since $null(Q_o)$ invariant under A, these columns are still in $null(Q_o)$ since $w_k \in null(Q_o)$ $k=1,\dots,n-r$

Then $P^{-1}AP = \begin{array}{c} \left[\begin{array}{c:c} \bar{A} & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ \hdashline ? & ? \end{array}\right] \begin{matrix} \}\,r \\ \}\,n-r \end{matrix} \\ \underbrace{\qquad}_{r}\ \underbrace{\quad}_{n-r} \end{array}$

Finally, let $P^{-1}B = \underbrace{\left[\begin{array}{c} \bar{B} \\ \hdashline ? \end{array}\right]}_{m} \begin{matrix} \}\,r \\ \}\,n-r \end{matrix}$

Then $\hat{G}(s) = C(sI-A)^{-1}B + D \overset{\text{by similarity transformation}}{=} CP(sI-P^{-1}AP)^{-1}P^{-1}B + D$

$$= [\underbrace{\bar{C}}_{r} \ \vdots\ 0] \left[\begin{array}{c:c} sI_r - \bar{A} & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ \hdashline \underbrace{?}_{r} & \underbrace{sI_{n-r} - ?}_{n-r} \end{array}\right]^{-1} \left[\begin{array}{c} \bar{B} \\ \hdashline 0 \cdots 0 \end{array}\right] \}\,r \ + D$$

$$= \bar{C}(sI_r - \bar{A})\bar{B} + D$$

set $\bar{D} = D$ and we see that $\{A, B, C, D\}$ <u>not</u> a minimal realization.

3. Chen 7.12

First system: $C(sI-A)^{-1}B = [2 \ \ 2] \begin{bmatrix} s-2 & -1 \\ 0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$= [2 \ \ 2] \cdot \frac{1}{(s-2)(s-1)} \begin{bmatrix} s-1 & 1 \\ 0 & s-2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{2(s-1)}{(s-2)(s-1)}$$

$$= \frac{2}{s-2} = \frac{2(s+1)}{(s-2)(s+1)} = \frac{2s+2}{s^2-s-2} \quad \text{(not minimal!)}$$

Second system: $C(sI-A)^{-1}B = [2 \ \ 0] \begin{bmatrix} s-2 & 0 \\ 1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$= [2 \ \ 0] \cdot \frac{1}{(s-2)(s+1)} \begin{bmatrix} s+1 & 0 \\ -1 & s-2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{2(s+1)}{(s-2)(s+1)} = \frac{2s+2}{s^2-s-2} \quad \text{(also not minimal!)}$$

both realizations are not minimal, moreover, since the evalues of the A matrices from system 1 and system 2 are not equal, there cannot exist P such that

$A_2 = P^{-1}A_1P$. Hence <u>not</u> algebraicly equivalent.

4. • Minimal realization first:

diff eq: $\dddot{y}(t) + y(t) = u(t)$, Let $x(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix}$

Then $$\dot{x} = \begin{bmatrix} \dot{y} \\ \ddot{y} \\ \dddot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u$$

The output equation follows immediately; $y = \underbrace{[1 \ 0 \ 0]}_{C} x$ with $D = 0$.

How do we know this is minimal? First, since $\hat{g}(s)$ is expressed with no common poles/zeros, then a minimal system will have dimension equal to the degree of the denominator. This is true here. Another check is to check if the system is <u>both</u> reachable & observable.

$$Q_r = [B \ \ AB \ \ A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \leftarrow \text{rank} = 3 \iff \text{reachable}$$

$$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \text{rank} = 3 \iff \text{observable.}$$

Hence system is minimal.

• Observable but not reachable now:
easiest way to do this is to add a common factor to the numerator & denominator of $\hat{g}(s)$ and then find observable canonical form.

$$\hat{g}(s) = \frac{1}{s^3+1} = \frac{s}{s^4+s}$$

Observable canonical form directly from $\hat{g}(s)$: (see Chen pp. 188)

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad C = [1 \ \ 0 \ \ 0 \ \ 0] \quad D = 0.$$

check observability: $$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{rank} = 4 \iff \text{observable}$$

not minimal + observable $\Rightarrow$ not reachable. Check:

$$Q_r = [B \ \ AB \ \ A^2B \ \ A^3B] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{rank} = 3 \iff \text{not reachable} \checkmark$$

- reachable but not observable now:

Same idea as before except use reachable / controllable canonical form:

$$A = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \qquad D = 0$$

$$Q_r = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{rank} = 4 \text{ reachable}$$

$$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \rightarrow \text{rank} = 3, \text{ not observable.}$$

5. a) Based on results from last homework, $\text{rank}(Q_r) = 1$ and since $n > 1$, this system is not reachable $\Rightarrow$ not minimal.

To find a minimal realization, we will take the existing realization & reduce it.

v is a basis for range Q_r. Let $P = [v \;\; w_1 \; \cdots \; w_{n-1}]$ where $w_1, \ldots, w_{n-1}$ are vectors in $\mathbb{R}^n$ chosen such that $\{v, w_1, \ldots, w_{n-1}\}$ form a linearly independent set. Then

$$AP = -vv^T [v \;\; w_1 \; \cdots \; w_{n-1}] = \begin{bmatrix} -\|v\|^2 v & \text{stuff} \end{bmatrix} =$$

$$= P \begin{array}{c} \left[\begin{array}{c|c} -\|v\|^2 & ? \\ \hline 0 & \\ \vdots & ? \\ 0 & \end{array}\right] \begin{array}{l} \}1 \\ \\ \}n-1 \\ \\ \end{array} \\ \underbrace{}_{1} \underbrace{}_{n-1} \end{array}$$

hence $$P^{-1}AP = \left[\begin{array}{c|c} -\|v\|^2 & ? \\ \hline 0 & \\ \vdots & ? \\ 0 & \end{array}\right] \qquad \text{let } \bar{A} = -\|v\|^2 \in \mathbb{R}$$

Since $B = v$, express $B = P \begin{bmatrix} \bar{B} \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \}1 \\ \\ \\ \\ \end{array}$ where $\bar{B} = 1$

Since $C = v^T$, $CP = [\underbrace{\|v\|^2}_{1} \; \vdots \; \underbrace{\text{stuff}}_{n-1}] \;\}1$

Let $D = 0$.

Claim that a minimal realization is

$$\bar{A} = -\|v\|^2$$
$$\bar{B} = 1$$
$$\bar{C} = \|v\|^2$$
$$\bar{D} = 0$$

by similarity transform.

proof: $C(sI_n - A)^{-1}B = CP(sI_n - P^{-1}AP)^{-1}P^{-1}B$

$$= \left[\bar{C} \;\vdots\; \text{stuff}\right] \left[\begin{array}{c|c} s+\|v\|^2 & ? \\ \hline 0 & \\ \vdots & sI_{n-1} - ? \\ 0 & \end{array}\right]^{-1} \left[\begin{array}{c} \bar{B} \\ \hline 0 \\ \vdots \\ 0 \end{array}\right]$$

using inverse of block diagonal matrix...

$$= \left[\bar{C} \;\vdots\; \text{stuff}\right] \left[\begin{array}{c|c} (s+\|v\|^2)^{-1} & ? \\ \hline 0 & \\ \vdots & ? \\ 0 & \end{array}\right] \left[\begin{array}{c} \bar{B} \\ \hline 0 \\ \vdots \\ 0 \end{array}\right]$$

$= \bar{C}(sI_1 - \bar{A})^{-1}\bar{B}$ ← shows that $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ indeed a realization.

Is it minimal? $Q_r = \bar{B} = 1$, rank $= 1 = n \Rightarrow$ reachable
$Q_o = \bar{C} = \|v\|^2$, since $v \neq 0$, rank$(Q_o) = 1 \Rightarrow$ observable
$\Rightarrow$ minimal.

b) The original system is not asymtotically stable because A has $n-1$ evalues equal to 0. A not Hurwitz $\Leftrightarrow$ system not A.S.

c) The new system is asymtotically stable.

$\bar{A} = -\|v\|^2$ has one "evalue" at $\lambda = -\|v\|^2 < 0$

$\bar{A}$ Hurwitz $\Rightarrow$ minimal system is A.S.

The point here is that if you have a minimal system that is A.S., non minimal realizations of this same system may not share this property since asymptotic stability describes internal behavior.

BIBO stability, however, is not affected by the minimality or nonminimality of the realization.

6. $\hat{G}(s) = \begin{bmatrix} \frac{s-1}{s} & 0 & \frac{s-2}{s+2} \\ 0 & \frac{s+1}{s} & 0 \end{bmatrix}$

Find McMillan degree:

First order minors: $\frac{s-1}{s}, \frac{s-2}{s+2}, \frac{s+1}{s}, 0$

Second order minors: $\frac{(s-1)(s+1)}{s^2}, 0, \frac{(s+1)(s-2)}{s(s+2)}$

Least common denominator: $s^2(s+2) \Rightarrow$ McMillan degree $= 3$.

Any valid realization of order 3 is then minimal. We just need to find A, B, C, D.

D is easy. $\lim_{s\to\infty} \hat{G}(s) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = D$

The strictly proper part of $\hat{G}(s)$ is then $\begin{bmatrix} \frac{-1}{s} & 0 & \frac{-4}{s+2} \\ 0 & 1/s & 0 \end{bmatrix} = C(sI-A)^{-1}B$

$B \in \mathbb{R}^{3\times 3}$
$A \in \mathbb{R}^{3\times 3}$
$C \in \mathbb{R}^{2\times 3}$

Lots of zeros in this TF so the best approach is to just realize each transfer function and then assemble the whole thing into a realization.

1. $y_1(s) = -\frac{1}{s} u_1(s) \Rightarrow \dot{x}_1 = u_1 \quad y = -x_1$
$y_1(s) = \frac{-2}{s+2} u_3(s) \Rightarrow \dot{x}_2 = -2x_2 + u_3 \quad y = -4x_2$

superposition:

$$y_1 = \begin{bmatrix} -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

check: $C(sI-A)^{-1}B = \begin{bmatrix} -1 & -4 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -1 & -4 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -4 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & 0 & 0 \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix} = \begin{bmatrix} \frac{-1}{s} & 0 & \frac{-2}{s+2} \end{bmatrix} \checkmark$$

Now $y_2(s) = \frac{1}{s} u_2(s) \Rightarrow \dot{x}_3 = u_2$, $y_2 = x_3$

put it all together:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$y = \begin{bmatrix} -1 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

checks with Matlab.

7. Want to find F such that $A-BF$ has evalues at -2 and $-1 \pm j$.

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} , \quad F = [f_1 \;\; f_2 \;\; f_3]$$

$$A-BF = \begin{bmatrix} 1-f_1 & 1-f_2 & -2-f_3 \\ 0 & 1 & 1 \\ -f_1 & -f_2 & 1-f_3 \end{bmatrix}$$

use direct method...

$$\det(\lambda I_3 - (A-BF)) = (\lambda+f_1-1)[(\lambda-1)(\lambda+f_3-1)+f_2] + f_1[(f_2-1)(-1) - (\lambda-1)(2+f_3)]$$
$$= \lambda^3 + 4\lambda^2 + 6\lambda + 4$$

Simplify... set powers equal.

$$\left.\begin{aligned} (f_3 + f_1 - 3)\lambda^2 &= 4\lambda^2 \\ (-4f_1 + f_2 - 2f_3 + 3)\lambda &= 6\lambda \\ (4f_1 - f_2 + f_3 - 1) &= 4 \end{aligned}\right\} \Rightarrow$$

unique solution

$F = [15 \;\; 47 \;\; -8]$

checks w/ Matlab.

8. Want to see what e-values can be achieved with state feedback.

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$\rightsquigarrow$ not controllable, can't use canonical form.

direct method possible but hard. Better idea is to use similarity transform to simplify the problem, then direct method.

Want $\bar{B} = P^{-1}B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ (less f_j's will appear in $\bar{A}-\bar{B}\bar{F}$)

can find this P^{-1} without even using Matlab, ... $P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

Then $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

$$\bar{A} = P^{-1}AP = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -3 & 0 & -1 \end{bmatrix}$$

hence
$$\left| \lambda I_4 - (\bar{A} - \bar{B}\bar{F}) \right| = \begin{vmatrix} \lambda-2 & -1 & 0 & 0 \\ \bar{f}_1 & \lambda-2+\bar{f}_2 & \bar{f}_3 & \bar{f}_4 \\ 0 & 0 & \lambda+1 & 0 \\ 0 & 3 & 0 & \lambda+1 \end{vmatrix}$$

$$= \lambda-2 \begin{vmatrix} \lambda-2+\bar{f}_2 & \bar{f}_3 & \bar{f}_4 \\ 0 & \lambda+1 & 0 \\ 3 & 0 & \lambda+1 \end{vmatrix} - \bar{f}_1 \begin{vmatrix} -1 & 0 & 0 \\ 0 & \lambda+1 & 0 \\ 3 & 0 & \lambda+1 \end{vmatrix}$$

$$= (\lambda-2)\left[(\lambda-2+\bar{f}_2)(\lambda+1)^2 + 3(-\bar{f}_4(\lambda+1)) \right] + \bar{f}_1 (\lambda+1)^2$$

$$= (\lambda+1)\left[(\lambda-2)(\lambda-2+\bar{f}_2)(\lambda+1) - 3\bar{f}_4(\lambda-2) + \bar{f}_1(\lambda+1) \right]$$

$$= (\lambda+1)\underbrace{\left[\lambda^3 + (f_2-3)\lambda^2 + (\bar{f}_2 - 3\bar{f}_4 + f_1)\lambda + 4 - 2\bar{f}_2 + 6\bar{f}_4 + \bar{f}_1 \right]}_{\text{selection of } F \text{ will allow me to place 3 e-values anywhere I want}}$$

however, I'm stuck with an e-value at -1. No choice of F will change this e-value.

This implies that we can find F that achieves the evalue sets $\{-2,-2,-1,-1\}$, $\{-2,-2,-2,-1\}$, but <u>not</u> $\{-2,-2,-2,-2\}$.

The system <u>is</u> thus stabilizable.