ECE504: Lecture 4

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29-Sep-2009
We are now starting Part II of ECE504: **Quantitative and qualitative analysis of systems**

mathematical description $\rightarrow$ results about behavior of system

Today:
1. Solution of LTI/LTV state equations for discrete-time systems
2. Solution of LTI/LTV state equations for continuous-time systems
3. Examples

You should be reading Chen Chapter 4 now. You should also refer back to Chen 3.2-3.3 to learn about “basis”, “linear independence”, and solutions to linear algebraic equations like $Ax = y$. 
Linear State-Space Description of Discrete-Time Systems

\[
x[k + 1] = A[k]x[k] + B[k]u[k]
\]
\[
y[k] = C[k]x[k] + D[k]u[k]
\]

We assume a general model with \( p \) inputs, \( q \) outputs, and \( n \) states.

Given an initial time \( k_0 \in \mathbb{Z} \), an initial state \( x[k_0] \in \mathbb{R}^n \), how does the state evolve for \( k = k_0 + 1, k_0 + 2, \ldots \)?
Solution to State Equation

Following our induction, for all \( k \geq k_0 \), we can write

\[
\mathbf{x}[k] = \Phi[k, k_0] \mathbf{x}[k_0] + \sum_{\ell=k_0}^{k-1} \Phi[k, \ell + 1] \mathbf{B}[\ell] \mathbf{u}[\ell]
\]

where \( \Phi \) is an \( n \times n \) matrix valued function with two time arguments:

\[
\Phi[k, j] = \begin{cases} 
\text{undefined} & k < j \\
I_n & k = j \\
\end{cases}
\]

Remarks:

- \( I_n \) is the \( n \times n \) identity matrix.
- The matrix function \( \Phi : \mathbb{Z}^2 \rightarrow \mathbb{R}^{n \times n} \) is called the state transition matrix (STM) corresponding to \( A[k] \).
Zero-Input Response

Recall that linear systems have the nice property that we can separately analyze the zero-input response and the zero-state response.

**Zero-input response:** Given \( u[k] = 0 \) for all \( k \geq k_0 \), we can write

\[
x[k] = \Phi[k, k_0]x[k_0]
\]

The state transition matrix \( \Phi[k, k_0] \) describes how the state at time \( k_0 \) evolves to the state at time \( k \geq k_0 \) (in the absence of an input).

If the STM \( \Phi[k, k_0] \) is invertible, then \( \Phi^{-1}[k, k_0] = \Phi[k_0, k] \). But there is no guarantee that it is invertible. This operation is one-way.
Zero-State Response

**Zero-state response**: Given $x[k_0] = 0$, we can write

$$x[k] = \sum_{\ell=k_0}^{k-1} \Phi[k, \ell + 1] B[\ell] u[\ell]$$

In this case, we have to compute several state transition matrices: $\Phi[k, k_0 + 1], \Phi[k, k_0 + 2], \ldots, \Phi[k, k]$.

This looks like it might require a lot of computation as $k$ gets larger. Fortunately, there are some nice properties of the state transition matrix that can ease the computational burden...
Some Basic Properties of the State Transition Matrix

1. $\Phi[j, j] = I_n$ for all $j \in \mathbb{Z}$.
2. $\Phi[k + 1, j] = A[k] \Phi[k, j]$ for all $k \geq j$.
3. If $\ell \leq j \leq k$, then $\Phi[k, \ell] = \Phi[k, j] \Phi[j, \ell]$.

This last property is called the “semigroup” property. It intuitively says that the transition from $x[\ell]$ to $x[k]$ is the same as the transition from $x[\ell]$ to $x[j]$ followed by the transition from $x[j]$ to $x[k]$. 

\[ x[\ell] \xrightarrow{\Phi[j, \ell]} x[j] \xrightarrow{\Phi[k, j]} x[k] \]
Special Case: $A[k] \equiv A$ for all $k \geq k_0$

When $A[k] \equiv A$ for all $k \geq k_0$, the product


How many $A$'s are involved in this product? 

Hence, when $A[k] \equiv A$ for all $k \geq k_0$, the state transition matrix can be written as

$$\Phi[k, j] = \begin{cases} 
\text{undefined} & k < j \\
I_n & k = j \\
A^{k-j} & k > j.
\end{cases}$$

In this case, the solution to the DT state-update difference equation is

$$x[k] = A^{k-k_0} x[k_0] + \sum_{\ell=k_0}^{k-1} A^{k-\ell-1} B[\ell] u[\ell]$$

for all $k \geq k_0$. 
Discrete-Time Output Solution

For all $k \geq k_0$, we can just plug our solution to the state equation into our state-space output equation to get

$$y[k] = C[k] \Phi[k, k_0] x[k_0] + C[k] \sum_{\ell=k_0}^{k-1} \Phi[k, \ell + 1] B[\ell] u[\ell] + D[k] u[k]$$

If the system is LTI, then we can write

$$y[k] = CA^{k-k_0} x[k_0] + C \sum_{\ell=k_0}^{k-1} A^{k-\ell-1} Bu[\ell] + Du[k]$$
Remarks on Discrete-Time State-Space Solutions

For causal, linear, lumped discrete-time systems with \( p \) input terminals, \( q \) output terminals, and \( n \) states, we have shown that, given \( x[k_0] \) and \( u[k] \) for all \( k \geq k_0 \), there exists a unique solution to the discrete-time state-update difference equation

\[
\]

That solution is

\[
x[k] = \Phi[k, k_0]x[k_0] + \sum_{\ell=k_0}^{k-1} \Phi[k, \ell + 1]B[\ell]u[\ell]
\]

for all \( k \geq k_0 \) with \( \Phi[k, j] \) as defined earlier.

This also implies that, given \( x[k_0] \) and \( u[k] \) for all \( k \geq k_0 \), there exists a unique solution to the discrete-time output equation.
Discrete-Time State-Space Example
Continuous-Time Linear Systems

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t) \]  \hspace{1cm} (1)

\[ y(t) = C(t)x(t) + D(t)u(t) \]  \hspace{1cm} (2)

**Theorem**

For any \( t_0 \in \mathbb{R} \), any \( x(t_0) \in \mathbb{R}^n \), and any \( u(t) \in \mathbb{R}^p \) for all \( t \geq t_0 \), there exists a unique solution \( x(t) \) for all \( t \in \mathbb{R} \) to the state-update differential equation (1). It is given as

\[ x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau) \, d\tau \quad t \in \mathbb{R} \]

where \( \Phi(t, s) : \mathbb{R}^2 \rightarrow \mathbb{R}^{n \times n} \) is the unique function satisfying

\[ \frac{d}{dt} \Phi(t, s) = A(t)\Phi(t, s) \text{ with } \Phi(s, s) = I_n. \]
Theorem Remarks

- Note that this theorem claims two things:
  1. A solution to the state-update equation always exists.
  2. The solution is unique.
- Does every differential equation have a unique solution?
  - What about
    \[ \dot{x}(t) = \frac{1}{t} \text{ with } x(0) = 5 \]
  - What about
    \[ \dot{x}(t) = 3(x(t))^{2/3} \text{ with } x(0) = 0 \]
- Proof sketch:
  1. Establish existence constructively by giving a solution and showing that it satisfies the state-update equation.
  2. Establish uniqueness by showing that, given two solutions to the state-update equation, they must be identical.
- We will only do the first part of the proof. Please refer to Chen or any other good textbook for a proof of the second part.
Theorem: Existence Proof Warmup #1

An important skill in research is to develop intuition by looking at the simplest possible case. What is the simplest possible case for the continuous-time state dynamics equation? Let’s first assume that everything is scalar, i.e. \( p = q = n = 1 \). Our state update equation becomes

\[
\dot{x}(t) = a(t)x(t) + b(t)u(t)
\]

Let

\[
\phi(t, s) := \exp \left\{ \int_s^t a(\tau) \, d\tau \right\}
\]

What is \( \phi(s, s) \)?

What is \( \frac{d}{dt} \phi(t, s) \)?
Theorem: Existence Proof Warmup #1

Note that $\phi(t, s) = \exp \left\{ \int_s^t a(\tau) d\tau \right\}$ always exists and satisfies its own differential equation:

$$\frac{d}{dt} \phi(t, s) = a(t)\phi(t, s) \text{ with } \phi(s, s) = 1.$$ 

Now let’s try the following solution to the scalar state-update differential equation with initial state condition $x(t_0)$:

$$x(t) = \phi(t, t_0)x(t_0) + \int_{t_0}^{t} \phi(t, \tau)b(\tau)u(\tau) d\tau \quad \forall t \in \mathbb{R}$$

To see that this is indeed a solution, we need to confirm two things:

1. Does our solution satisfy the initial condition requirement of the scalar state-update DE?
2. Does our solution really solve the scalar state-update DE?
To develop additional intuition, let’s now assume that everything is time-invariant, i.e. $A(t) \equiv A$ and $B(t) \equiv B$. Our state update equation becomes

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Let

$$\Phi(t, s) := \sum_{k=0}^{\infty} A^k \frac{1}{k!} (t - s)^k$$

What is $\Phi(s, s)$?

What is $\frac{d}{dt} \Phi(t, s)$?
Theorem: Existence Proof Warmup #2

Note that $\Phi(t, s) = \sum_{k=0}^{\infty} A^k \frac{1}{k!} (t - s)^k$ exists for any $A \in \mathbb{R}^{n \times n}$, $t \in \mathbb{R}$, and $s \in \mathbb{R}$. Moreover, $\Phi(t, s)$ satisfies its own differential equation:

$$\frac{d}{dt} \Phi(t, s) = A \Phi(t, s) \text{ with } \Phi(s, s) = I_n.$$

Now let’s try the following solution to the time-invariant matrix state-update differential equation with initial state condition $x(t_0)$:

$$x(t) = \Phi(t, t_0) x(t_0) + \int_{t_0}^{t} \Phi(t, \tau) B u(\tau) d\tau \quad \forall t \in \mathbb{R}$$

To see that this is indeed a solution, we need to confirm two things:

1. Does our solution satisfy the initial condition requirement of the time-invariant matrix state-update DE?
2. Does our solution really solve the time-inv. matrix state-update DE?
Theorem: Existence Proof for General Case

For the general (non-scalar, time-varying) case, we propose the solution

\[ \mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \Phi(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau \]  

(3)

where the state transition matrix satisfies the matrix differential equation

\[ \frac{d}{dt} \Phi(t, s) = A(t)\Phi(t, s) \text{ with } \Phi(s, s) = I_n. \]  

(4)

Note that (4) is consistent with our two warmup cases.

To complete the existence proof, we need to:

1. Show that (3) with \( \Phi(t, s) \) defined according to (4) satisfies the initial condition requirement of the state-update DE.
2. Show that (3) with \( \Phi(t, s) \) defined according to (4) is indeed a solution to the state-update DE.
3. Show that there always exists a solution to the matrix DE (4).
Show that \((3)\) with \(\Phi(t,s)\) defined according to \((4)\) satisfies the initial condition requirement of the state-update DE.
Theorem: Existence Proof for General Case: Part 2

Show that (3) with $\Phi(t, s)$ defined according to (4) is indeed a solution to the state-update DE.
Show that there always exists a solution to the matrix DE (4).
Peano-Baker Series Example

\[ A(t) = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} \]
Fundamental Matrix Method

While the Peano-Baker series establishes existence (and thus concludes the proof of the existence and uniqueness theorem), it is sometimes easier to find $\Phi(t, s)$ via the “fundamental matrix method” (Chen section 4.5).

Basic idea:

1. Consider the the continuous time DE with $\mathbf{x}(t) \in \mathbb{R}^n$

   $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t)$  \hspace{1cm} (5)

2. Choose $n$ different initial conditions $\mathbf{x}_1(t_0), \ldots, \mathbf{x}_n(t_0)$. These $n$ initial condition vectors must be linearly independent.

3. These $n$ different initial conditions lead to $n$ different solutions to (5). Call these solutions $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ and put them into a matrix $\mathbf{X}(t) = [\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)] \in \mathbb{R}^{n \times n}$.

4. Note that $\dot{\mathbf{X}}(t) = A(t)\mathbf{X}(t)$. The quantity $\mathbf{X}(t)$ is called a fundamental matrix of (5). Is the fundamental matrix unique?
Let $X(t)$ be any fundamental matrix of (5). Note that $X(t)$ is invertible for all $t$ (see Chen p. 107). The state transition matrix $\Phi(t, s)$ can then be computed as

$$\Phi(t, s) = X(t)X^{-1}(s).$$

Check:

$$\Phi(s, s) =$$

$$\frac{d}{dt} \Phi(t, s) =$$
Fundamental Matrix Example

\[ A(t) = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} \]
Remarks on the CT State-Transition Matrix $\Phi(t, s)$

1. There are many ways to compute $\Phi(t, s)$. Some are easier than others, but computing $\Phi(t, s)$ is almost always difficult.

2. Do different methods for computing $\Phi(t, s)$ lead to different solutions?

3. Unlike the DT-STM $\Phi[k, j]$, the CT-STM $\Phi(t, s)$ is defined for any $(t, s) \in \mathbb{R}^2$. This means that we can specify an initial state $x(t_0)$ and compute the system response at times prior to $t_0$.

4. It is easy to show that $\Phi(t, s)$ possesses the semi-group property, i.e.

$$\Phi(t, \tau) = \Phi(t, s) \Phi(s, \tau)$$

for any $(t, \tau, s) \in \mathbb{R}^3$ from the fundamental matrix formulation:

$$\Phi(t, \tau) = \Phi(t, s) \Phi(s, \tau) = X(t)X^{-1}(s)X(s)X^{-1}(\tau) = X(t)X^{-1}(\tau)$$
Important Special Case: \( A(t) \equiv A \)

When \( A(t) \equiv A \), the state-transition matrix Peano-Baker series becomes:

\[
\Phi(t, s) = \sum_{k=0}^{\infty} M_k(t, s)
\]

\[
= \sum_{k=0}^{\infty} \int_t^s \int_s^{\tau_1} \cdots \int_s^{\tau_{k-1}} A A \cdots A d\tau_k \cdots d\tau_1
\]

\[
= \sum_{k=0}^{\infty} A^k \int_t^s \int_s^{\tau_1} \cdots \int_s^{\tau_{k-1}} d\tau_k \cdots d\tau_1
\]

To compute \( M_k(t, s) \), let’s look at \( k = 0, 1, 2, \ldots \) to see the pattern...
Important Special Case: $A(t) \equiv A$

By induction, we can show that

$$M_k(t, s) = A^k \frac{1}{k!} (t - s)^k$$

hence

$$\Phi(t, s) = \sum_{k=0}^{\infty} A^k \frac{1}{k!} (t - s)^k$$

which is consistent with our earlier result (warmup #2).

Suppose, for $x \in \mathbb{C}$, we have

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

What is $f(x)$?
Definition (Matrix Exponential)

Given \(W \in \mathbb{C}^{n \times n}\), the matrix exponential is defined as

\[
\exp(W) = \sum_{k=0}^{\infty} \frac{W^k}{k!}
\]

Note that the matrix exponential is not performed element-by-element, i.e.

\[
\exp \left( \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \right) \neq \begin{bmatrix} e^{w_{11}} & e^{w_{12}} \\ e^{w_{21}} & e^{w_{22}} \end{bmatrix}
\]

Matlab has a special function (\texttt{expm}) that computes matrix exponentials. Calling \(\exp(W)\) will not give the same results as \(\texttt{expm}(W)\).
Important Special Case: $A(t) \equiv A$

Putting it all together, when $A(t) \equiv A$, we can say that

$$
\Phi(t, s) = \sum_{k=0}^{\infty} A^k \frac{1}{k!} (t - s)^k = \exp \{(t - s)A\}
$$

Then the solution to the LTI continuous-time state-update DE is

$$
x(t) = \exp \{(t - t_0)A\} x(t_0) + \int_{t_0}^{t} \exp \{(t - \tau)A\} B(\tau)u(\tau) \, d\tau
$$

and the output equation is

$$
y(t) = C(t) \exp \{(t - t_0)A\} x(t_0) + C(t) \int_{t_0}^{t} \exp \{(t - \tau)A\} B(\tau)u(\tau) \, d\tau + D(t)u(t)
$$
Contrast/Comparison Between CT and DT Solutions

Similarities

- CT and DT solutions have same “look”.
- CT and DT solutions have state transition matrices with same intuitive properties, e.g. semigroup.

Differences

- In DT systems, $x[k]$ is only defined for $k \geq k_0$ because the DT-STM $\Phi[k, k_0]$ is only defined for $k \geq k_0$.
- In CT systems, $x(t)$ is only defined for all $t \in \mathbb{R}$ because the CT-STM $\Phi(t, t_0)$ is defined for all $(t, t_0) \in \mathbb{R}^2$.
- We didn’t prove this, but the CT-STM $\Phi(t, t_0)$ is always invertible. This is not true of the DT-STM $\Phi[k, k_0]$. 
Conclusions: What We Now Know

- We know how to solve discrete-time LTV and LTI systems. “Solve” means “write an analytical expression for $x[k]$ and $y[k]$ given $A[k]$, $B[k]$, $C[k]$, $D[k]$, and $x[k_0]$.”
- We know that solutions must exist and must be unique.

- We know how to solve continuous-time LTV and LTI systems. “Solve” means “write an analytical expression for $x(t)$ and $y(t)$ given $A(t)$, $B(t)$, $C(t)$, $D(t)$, and $x(t_0)$.”
- We know that solutions must exist and must be unique.
- We also know two ways to compute the state transition matrix.

- We know some of the properties of state transition matrices.
- We know differences between the DT-STM and the CT-STM.
Next Time

1. Linear algebra tools to lay foundation for analysis of $A^k$ and $\exp(A)$:
   - Subspaces
   - Nullspace and range
   - Rank
   - Matrix invertibility equivalences
2. Efficient ways to analyze and compute $A^k$ and $\exp(A)$.
3. More DT-LTI and CT-LTI examples.