1. Chan 5.10

\[ X = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

Compute eigenvalues...

\[ \det(\lambda I - A) = \det \begin{bmatrix} \lambda + 1 & 0 & -1 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = (\lambda + 1)\lambda^2 \]

\[ \lambda_1 = -1, \quad r_1 = 1 \]
\[ \lambda_2 = 0, \quad r_2 = 2 \rightarrow \text{not asymptotically stable} \]

So we need to check the geometric multiplicity of \( \lambda_2 \) to check (marginal) stability...

\[ m_2 = \dim (\text{null}(\lambda_2 I - A)) = \dim (\text{null}(A)) = \dim (\text{null}(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix})) = 2 \]

E.g., \( v_{21} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \) and \( v_{22} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \)

are linearly independent and both in \( \text{null}(\lambda_2 I - A) \).

Hence, according to the theorem presented in lecture, this system is (marginally) stable.
2. Chen 5.13

\[ x_{k+1} = \begin{bmatrix} 0.9 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x_k \]

compute e-values: \( \lambda_1 = 0.9 \), \( r_1 = 1 \)
\( \lambda_2 = 1 \), \( r_2 = 2 \)

\( \text{not asymptotically stable} \)

Need to check geometric multiplicity of \( \lambda_2 \) to see if this system is (marginally) stable...

\[ m_2 = \dim(\text{null}(\lambda_2 - A)) = \dim(\text{null}\left(\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)) = 1 \]

only one linearly independent vector in the nullspace.

hence \( m_2 = 1 < r_2 = 2 \) \( \Rightarrow \text{not marginally stable} \)

3. Chen 5.18

This is an if and only if theorem, so we will first assume the equation

\[ A^T M + M A + 2 \mu M = -N \] (1)

has a unique positive definite solution for \( M \), given any positive definite matrix \( N \).

Rewrite (1)...

\[ A^T M + M A + \mu I M + \mu M I = -N \]

\[ (A^T + \mu I)M + M (A + \mu I) = -N \] (2)

The Lyapunov theorem states that if \( N \) is positive definite and there is a unique positive definite solution for \( M \) in (2), then the e-values of \( A + \mu I \) must all have real parts less than zero. Continued...
But how do the e-values of \( A+MI \) relate to the e-values of \( A \)?

If \( v_i \) is an e-vector associated with e-value \( \lambda_i \), then we know that

\[
A v_i = \lambda_i v_i
\]

This implies that

\[
(A+MI) v_i = Av_i + Mu_i = (\lambda_i + M) v_i.
\]

Hence \( v_i \) is also an e-vector of \( A+MI \) and has the associated e-value \( \gamma_i = \lambda_i + M \).

Our previous result implied that \( \Re \{ \gamma_i \} < 0 \) for all \( i \)

hence \( \Re \{ \lambda_i + M \} < 0 \) for all \( i \)

hence \( \Re \{ \gamma_i \} < -M \) for all \( i \)

which is the result we wanted (all e-values of \( A \) have real part less than \(-M\)).

Now, if we assume all of the e-values of \( A \) have real part less than \(-M\), then all of the e-values of \( A+MI \) must have real part less than zero.

The Lyapunov stability theorem is an if and only if theorem, so if \( A+MI \) has all evales with real part less than zero, then

\[
(A+MI)^T M + M (A+MI) = -N
\]

has a unique pos. def. solution for \( M \) given any choice of positive def. But this can be rewritten as

\[
A^T M + MA + 2MU = -N,
\]

which is the result we wanted.
4. Chen 5.23

You can use the fundamental matrix method to get

$$
\Phi(t,s) = \begin{bmatrix}
e^{-(t-s)} & 0 \\
0.2(e^{-5(t-s)} - e^{-4s}) & 1
\end{bmatrix}
$$

given any $s \geq 0$, the elements of $\Phi(t,s)$ are all bounded for all $t \geq s$. So this system is marginally stable.

To check asymptotic stability, let $t \to \infty$ (fix $s$)

$$
\lim_{t \to \infty} \Phi(t,s) = \begin{bmatrix}
0 & 0 \\
-0.2e^{-4s} & 1
\end{bmatrix}
$$

Note that some of the terms in the STM do not go to zero, hence this system is not asymptotically stable.
(a) For diagonal $A$ we have

$$A = \begin{pmatrix} \lambda_1 & \cdots & \lambda_n \end{pmatrix} = \Lambda$$

so the Lyapunov equation is

$$\Lambda P + P \Lambda = -Q$$

thus

$$\lambda_i p_{ij} + p_{ij} \lambda_j = -q_{ij}$$

and

$$p_{ij} = -\frac{q_{ij}}{\lambda_i + \lambda_j}$$

(b) With $A$ diagonalizable

$$A = T \Lambda T^{-1}$$

where

$$T = (q_1 \cdots q_n)$$

and $q_i$ is the eigenvector of $A$ corresponding to $\lambda_i$. Thus the Lyapunov equation can be written as

$$(T \Lambda T^{-1})^T P + P (T \Lambda T^{-1}) = -Q$$

or

$$\Lambda (T^T P T) + (T^T P T) \Lambda = -T^T Q T$$

Let $\tilde{P} = T^T P T$ and $\tilde{Q} = T^T Q T$ then

$$\Lambda \tilde{P} + \tilde{P} \Lambda = -\tilde{Q}$$

and

$$\tilde{p}_{ij} = -\frac{\tilde{q}_{ij}}{\lambda_i + \lambda_j}$$

$$P = (T^{-1})^T \tilde{P} T^{-1}$$