Distributed Reception with Hard Decision Exchanges

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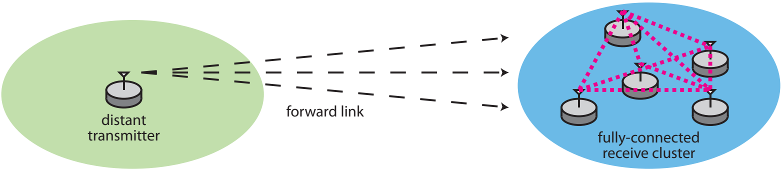
System Model



Motivation:

- Diversity gain
- Receive beamformer emulation: N-fold SNR gain
- Implementation possible with off-the-shelf hardware, e.g. WiFi LAN
- No loss of rate if LAN is on separate channel from forward link
- Robustness (no single point of failure)

Distributed Beamforming Throughput Example

Direct emulation of a receive beamformer:

Forward Link Bit Rate (info bits / sec)	÷	Code Rate (info bits / code bits)	÷	Modulation (code bits / symbol)	×	Quantization (LAN bits / observation)	×	N receivers (# receivers)	=	LAN Throughbut (LAN bits / sec)
1Mbps	÷	1/2	÷	1	×	16	×	10	=	320Mbps

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Distributed reception with hard decision exchanges:

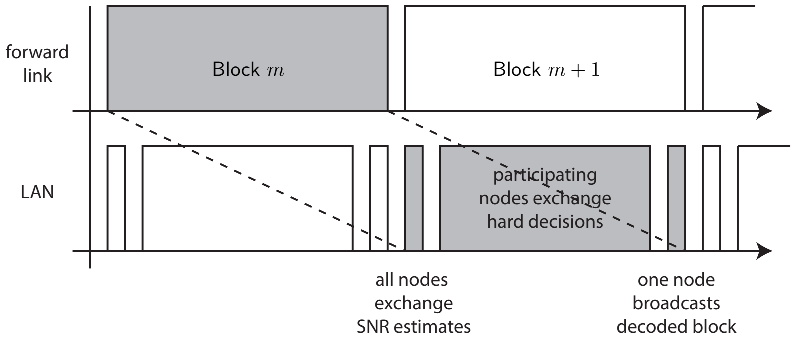
Forward Link Bit Rate (info bits / sec)	÷	Code Rate (info bits / code bits)	÷	Modulation (code bits / symbol)	×	Hard-Decision Quantization (LAN bits / symbol)	×	N receivers (# receivers)	=	LAN Throughbut (LAN bits / sec)
1Mbps	÷	1/2	÷	1	×	1	×	10	=	20Mbps

Assumptions

System model assumptions:

- Forward link channels are AWGN (no interference).
- Forward link messages are (n, k) block encoded at rate $r = \frac{k}{n}$
- Forward link is block i.i.d. fading.
- LAN supports broadcast transmission.
- LAN communication is reliable.

Protocol and Throughput Requirements



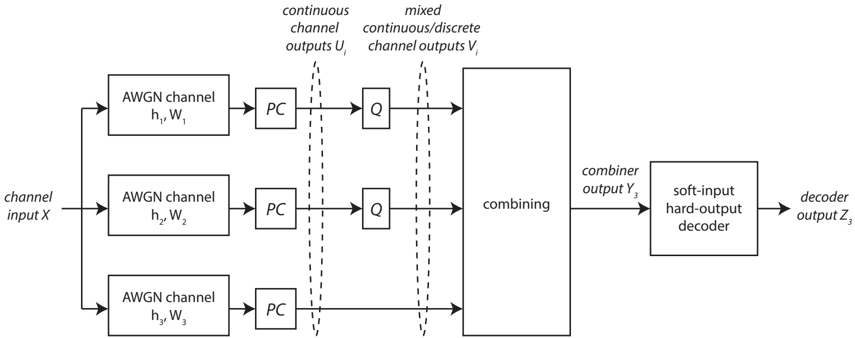
Normalized LAN throughput (bits per forward link information bit):

$$\eta_{\text{LAN}} = \frac{No_1 + Kn + k + o_2}{k} \approx \frac{K}{r} + 1$$

where $K \leq N$ is the number of "participating" nodes selected so that

$$K \leq \min\{N, r(C_{\text{LAN}} - 1)\}$$

Combining



Notation:

- Forward link symbols $X \in \mathcal{X}$ (assumed to be equiprobable).
- $U_i = |h_i|X + W_i$ is the phase-corrected soft output at node i.
- $V_i \in \mathcal{X}$ is the hard decision at node i.
- Y_i is the combiner output at node i.

Benchmark: Ideal Receive Beamforming

Combiner output:

$$\begin{aligned} Y_{\mathrm{bf}} \equiv Y_i &= \sum_{j \in \mathcal{P}} |h_j| U_j \\ &= \sum_{j \in \mathcal{P}} |h_j|^2 X + \sum_{j \in \mathcal{P}} |h_j| W_j \\ &= \|\boldsymbol{h}_{\mathcal{P}}\|^2 X + \tilde{W} \end{aligned}$$

where $\mathcal{P} \subseteq \{1, \dots, N\}$ is the set of participating nodes, $\boldsymbol{h}_{\mathcal{P}} \in \mathbb{C}^K$ is the channel vector corresponding to the participating nodes, and $\tilde{W} \sim \mathcal{CN}(0, N_0 \|\boldsymbol{h}_{\mathcal{P}}\|^2)$.

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$Y_{\rm bf}$ is conditionally Gaussian with mean $\mathrm{E}[Y_{\rm bf}|X] = \|\boldsymbol{h}_{\mathcal{P}}\|^2 X$ and variance $\mathrm{var}[Y_{\rm bf}|X] = N_0 \|\boldsymbol{h}_{\mathcal{P}}\|^2$. The SNR of ideal receive beamforming is then

$$
\mathrm{SNR}_{\rm bf} = \begin{cases} \dfrac{\mathrm{E}\left\{|\mathrm{E}[Y_i|X]|^2\right\}}{\mathrm{var}[Y_i]} = \dfrac{\|\boldsymbol{h}_{\mathcal{P}}\|^2 \mathcal{E}_s}{N_0} & \text{(complex alphabets)} \\ \dfrac{\mathrm{E}\left\{(\mathrm{E}[\mathrm{Re}(Y_i)|X])^2\right\}}{\mathrm{var}[\mathrm{Re}(Y_i)]} = \dfrac{\|\boldsymbol{h}_{\mathcal{P}}\|^2 \mathcal{E}_s}{N_0/2} & \text{(real alphabets)} \end{cases}
$$

Idea 1: Pseudobeamforming (orig: Matt Rebholz BBN)

Combiner output:

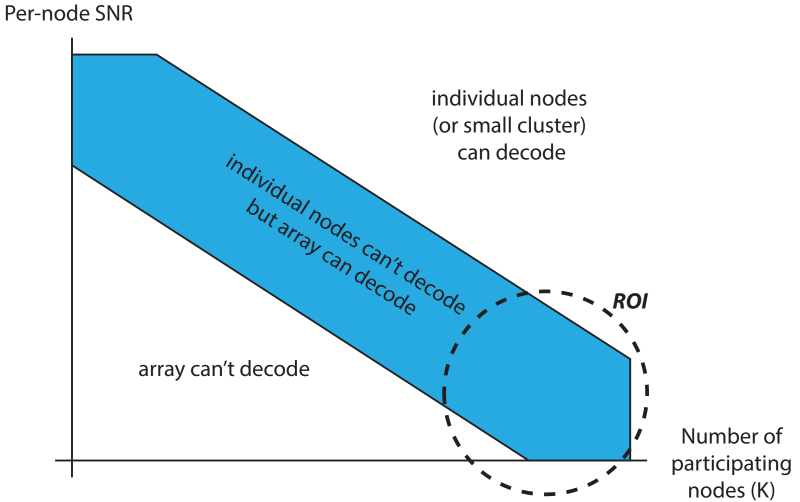
$$Y_{\text{pbf}} \equiv Y_i = \sum_{j \in \mathcal{P}} |h_j| V_j.$$

Remarks:

1. Just like beamforming, except we are summing weighted **hard decisions**.
2. We are ignoring local soft information.
3. Clearly suboptimal, but conceptually straightforward.

Question: How well does pseudobeamforming perform with respect to ideal receive beamforming?

Regime of Interest



Asymptotic Gaussianity of Pseudobeamforming

Theorem

Under certain regularity conditions on the channels, as $|\mathcal{P}| \to \infty$, *we have*

$$A = \frac{\mathrm{Re}(Y_{\mathrm{pbf}}) - \sum_{j\in\mathcal{P}} |h_j| \mathrm{E}[\mathrm{Re}(V_j) \,|\, X]}{\sqrt{\sum_{j\in\mathcal{P}} |h_j|^2 \mathrm{var}[\mathrm{Re}(V_j) \,|\, X]}} \xrightarrow{d} \mathcal{N}(0,1) \quad (1)$$

$$B = \frac{\mathrm{Im}(Y_{\mathrm{pbf}}) - \sum_{j\in\mathcal{P}} |h_j| \mathrm{E}[\mathrm{Im}(V_j) \,|\, X]}{\sqrt{\sum_{j\in\mathcal{P}} |h_j|^2 \mathrm{var}[\mathrm{Im}(V_j) \,|\, X]}} \xrightarrow{d} \mathcal{N}(0,1) \quad (2)$$

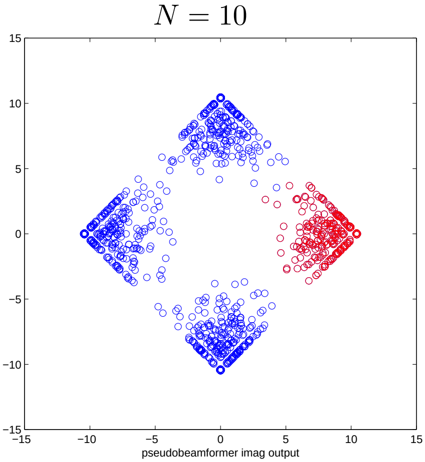
when conditioned on X *and* $\boldsymbol{h}$ *where* $\xrightarrow{d}$ *means convergence in distribution.*

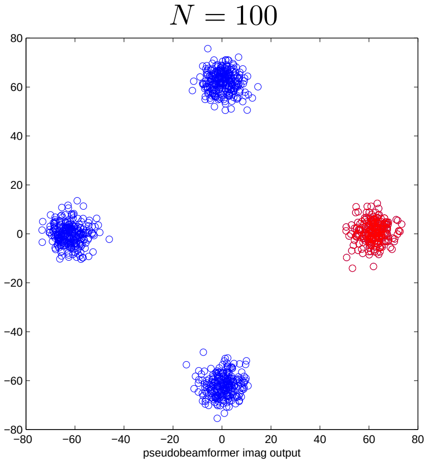
Recall the pseudobeamformer combiner output:

$$Y_{\mathrm{pbf}} = \sum_{j\in\mathcal{P}} |h_j| V_j.$$

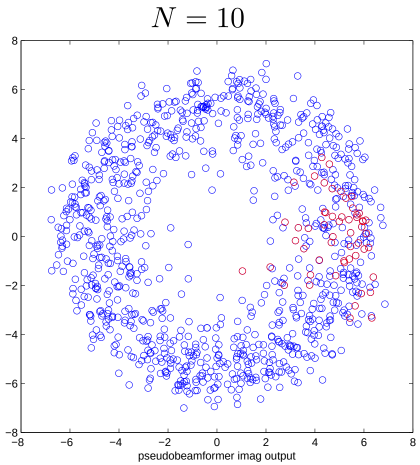
Since the $\{V_j\}$ are cond. independent but not identically distributed, the proof requires the use of the Lindeberg central limit theorem.

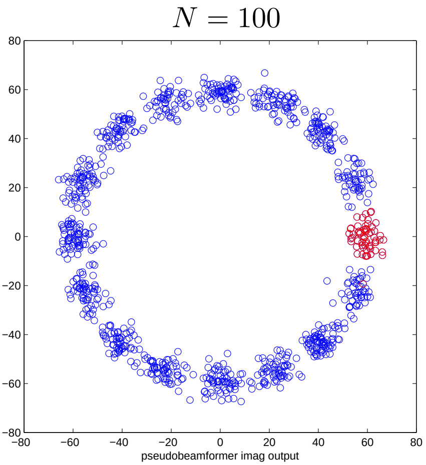
QPSK Pseudobeamforming Examples





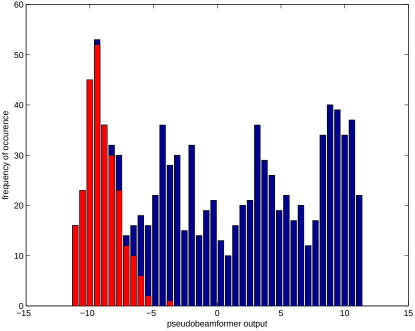
16PSK Pseudobeamforming Examples



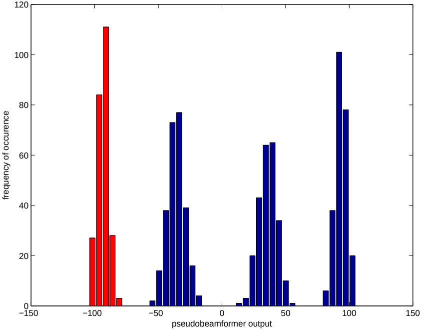


4PAM Pseudobeamforming Examples

$N = 10$



$N = 100$



Low per-node SNR Statistics of M-PAM

Theorem

For M-PAM forward link modulation with equiprobable symbols and alphabet $\mathcal{X} = \{x_1, \dots, x_M\} = \{(-M+1)a, \dots, (M-1)a\}$ in the low per-node SNR regime, we have

$$\mathrm{E}[\mathrm{Re}(V_j) \,|\, X = x_\ell] \approx \left(\frac{2(M-1)\rho_j}{\sqrt{2\pi}} \right) x_\ell$$

$$\mathrm{var}[\mathrm{Re}(V_j) \,|\, X = x_\ell] \approx (M-1)^2 a^2$$

for all $\ell \in \{1, \dots, M\}$ where $\rho_j^2 := \frac{|h_j|^2 a^2}{N_0/2}$.

The proof uses a low-SNR Q-function approximation:

$$Q(x) = \frac{1}{2} - \int_0^x \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt \approx \frac{1}{2} - \frac{x}{\sqrt{2\pi}}.$$

Asymptotic Performance of M-PAM Pseudobeamforming

Corollary

Given M-PAM forward link modulation with equiprobable symbols. In the low per-node SNR regime as $K \to \infty$ and assuming channel regularity, we have

$$\mathrm{SNR}_{\mathrm{pbf}}^{M-\mathrm{PAM}} \approx \frac{2}{\pi}\mathrm{SNR}_{\mathrm{bf}}.$$

From the prior results, can show the conditional M-PAM distribution follows

$$Y_{\mathrm{pbf}} \sim \mathcal{N}\left(\frac{2a(M-1)}{\sqrt{N_0\pi}}x_\ell\|\boldsymbol{h}_{\mathcal{P}}\|^2, (M-1)^2a^2\|\boldsymbol{h}_{\mathcal{P}}\|^2\right)$$

Some straightforward algebra then yields the desired result.

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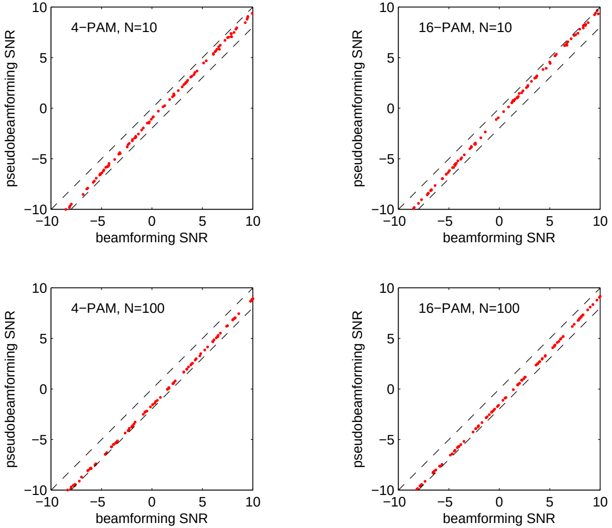
Some straightforward algebra then yields the desired result.

This result includes BPSK and can be easily extended to M^2-QAM and QPSK. For these modulation formats, we can expect pseudobeamforming to have a $10\log_{10}(2/\pi) \approx -1.96$ dB loss wrt ideal receive beamforming.

M-PAM Pseudobeamforming SNR vs. Beamforming SNR



Low per-node SNR Statistics of M-PSK

Theorem

For M-PSK forward link modulation with $M \geq 4$, M even, equiprobable symbols drawn from the alphabet $\mathcal{X} = \{x_1, \dots, x_M\}$ with $x_m = ae^{j2\pi(m-1)/M}$, and in the low per-node SNR regime, we have

$$\mathrm{E}[V_j \,|\, X = x_\ell] \approx \left(\frac{M \rho_j \sin(\pi/M)}{2\sqrt{\pi}} \right) x_\ell$$

where $\rho_j^2 := \frac{|h_j|^2 a^2}{N_0}$ *and*

$$\mathrm{var}[V_j \,|\, X = x_\ell] \approx a^2$$

for all $\ell \in \{1, \dots, M\}$. Moreover, in the low per-node SNR regime, $\mathrm{var}[\mathrm{Re}(V_j) \,|\, X = x_\ell] \approx \frac{a^2}{2}$, $\mathrm{var}[\mathrm{Im}(V_j) \,|\, X = x_\ell] \approx \frac{a^2}{2}$ *and* $\mathrm{cov}[\mathrm{Re}(V_j), \mathrm{Im}(V_j) \,|\, X = x_\ell] \approx 0$ *for all $\ell \in \{1, \dots, M\}$.*

The proof uses a first-order Taylor series approximation to express the hard decision probabilities at low per-node SNR.

Asymptotic Performance of M-PSK Pseudobeamforming

Corollary

Given M-PSK forward link modulation with equiprobable symbols and alphabet $\mathcal{X} = \{x_1, \dots, x_M\} = \left\{a, ae^{j2\pi/M}, ae^{j4\pi/M}, \dots, ae^{j(M-1)2\pi/M}\right\}$. In the low per-node SNR regime as $K \to \infty$ and assuming channel regularity, we have

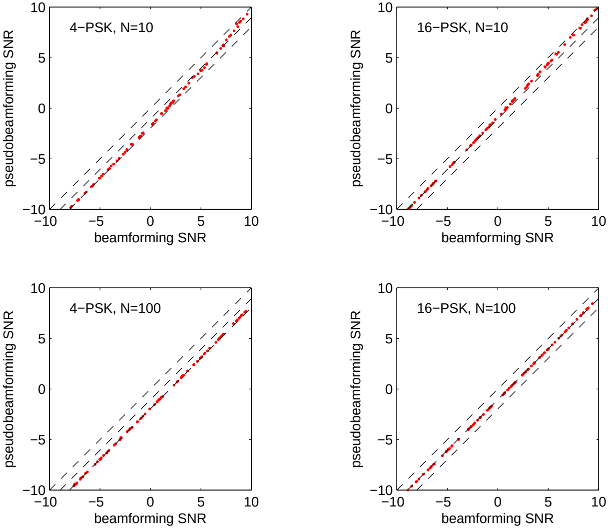
$$\mathrm{SNR}_{\mathrm{pbf}}^{M-\mathrm{PSK}} = \frac{M^2 \sin^2(\pi/M)}{4\pi} \mathrm{SNR}_{\mathrm{bf}}.$$

and

$$\lim_{M \to \infty} \mathrm{SNR}_{\mathrm{pbf}}^{M-\mathrm{PSK}} \approx \frac{\pi}{4} \mathrm{SNR}_{\mathrm{bf}}$$

- The function $\frac{M^2 \sin^2(\pi/M)}{4\pi}$ is decreasing in M.
- For $M = 4$, we have $\frac{M^2 \sin^2(\pi/M)}{4\pi} = \frac{2}{\pi}$ (-1.96 dB, consistent with M-PAM).
- In the limit as $M \to \infty$ we can use a small angle approximation to compute $\frac{M^2 \sin^2(\pi/M)}{4\pi} \to \frac{\pi}{4}$ (-1.05dB).

M-PSK Pseudobeamforming SNR vs. Beamforming SNR



Idea 2: Optimal Combining

Basic idea:

- Use all of the available information at each receiver (including the unquantized local observation)
- Produce correct posterior probability for each transmitted symbol based on the mixed continuous/discrete vector observation
- These posteriors can then be converted to LLRs for input to the soft-input decoder

Block error rate performance will be better than pseudobeamforming.

Optimal Combining: Computing Posteriors

The posterior probability of symbol $X = x_m \in \mathcal{X}$ given the vector observation $\boldsymbol{V}$ can be written as

$$\begin{aligned}\mathrm{Prob}(X = x_m | \boldsymbol{V} = \boldsymbol{v}) &= \frac{p_{\boldsymbol{V}|X}(\boldsymbol{v}|X = x_m)\mathrm{Prob}(X = x_m)}{p_{\boldsymbol{V}}(\boldsymbol{v})} \\ &= \frac{\prod_{i=1}^{N} p_{V_i|X}(v_i|X = x_m)}{\sum_{\ell=1}^{M} \prod_{i=1}^{N} p_{V_i|X}(v_i|X = x_\ell)}\end{aligned}$$

where the second equality uses the equiprobable symbol assumption and the fact that the elements of $\boldsymbol{V}$ are conditionally independent.

Each receive node must compute $p_{V_i|X}(v_i|X = x_\ell)$ for all $i = 1, \dots, N$ and $\ell = 1, \dots, M$. This is possible since the channel magnitudes $\{|h_1|, \dots, |h_N|\}$ are known to all of the nodes in the receive cluster.

Optimal Combining: The Details

Suppose you are receive node j. Since your local observation is unquantized, we have $v_j = u_j$ and

$$p_{V_j|X}(v_j|X = x_\ell) = \begin{cases} \frac{1}{\pi N_0} \exp\left(-\frac{|v_j - |h_j| x_\ell|^2}{N_0}\right) & \text{(complex alphabets)} \\ \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(v_j - |h_j| x_\ell)^2}{N_0/2}\right) & \text{(real alphabets)} \end{cases}$$

for $\ell = 1, \dots, M$ and any forward link modulation.

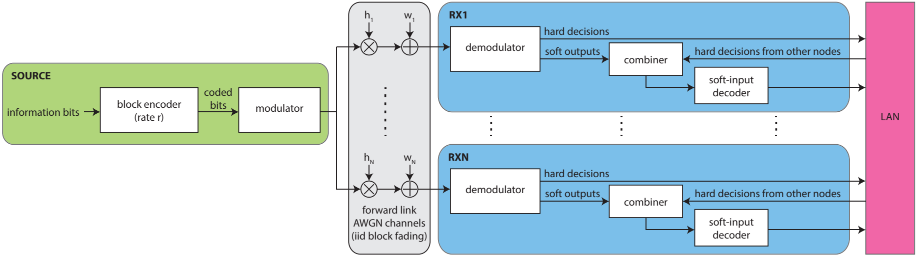
The remaining $p_{V_i|X}(v_i|X = x_\ell)$ for $i \neq j$ are the channel transition probabilities of the discrete memoryless channel (DMC) induced by the hard decision at node i. For example, with a BPSK forward link with alphabet $\mathcal{X} = \{x_1, x_2\}$, we have

$$p_{V_i|X}(v_i = x_m|X = x_\ell) = \begin{cases} 1 - p & m = \ell \\ p & m \neq \ell \end{cases}$$

with crossover probability

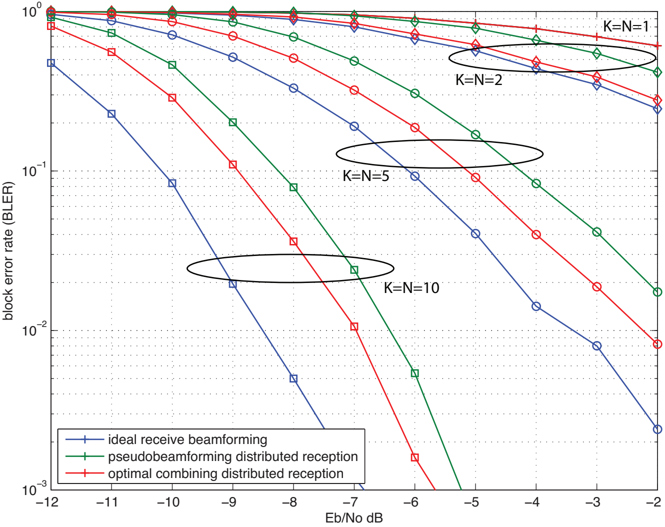
$$p = Q\left(|h_i| \sqrt{\frac{2\mathcal{E}_s}{N_0}}\right).$$

Numerical Results: Simulation Parameters

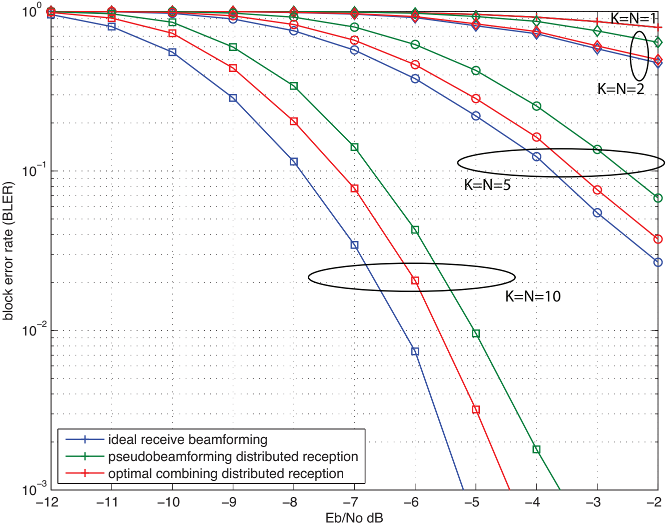


- Block fading scenario.
- Channel assumed to be $h_j[m] \overset{\text{i.i.d.}}{\sim} \mathcal{CN}(0,1)$.
- $(8100, 4050)$ rate 1/2 LDPC code for DVB-S2.
- Results averaged over 5000 channel/noise realizations.

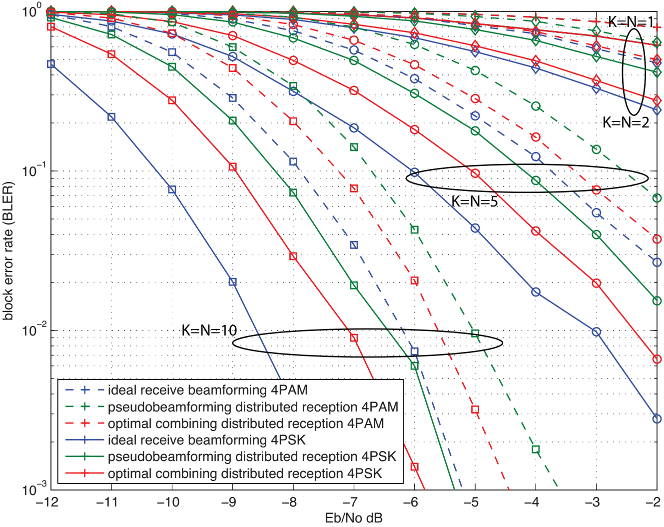
Block Error Rate Performance: BPSK/QPSK



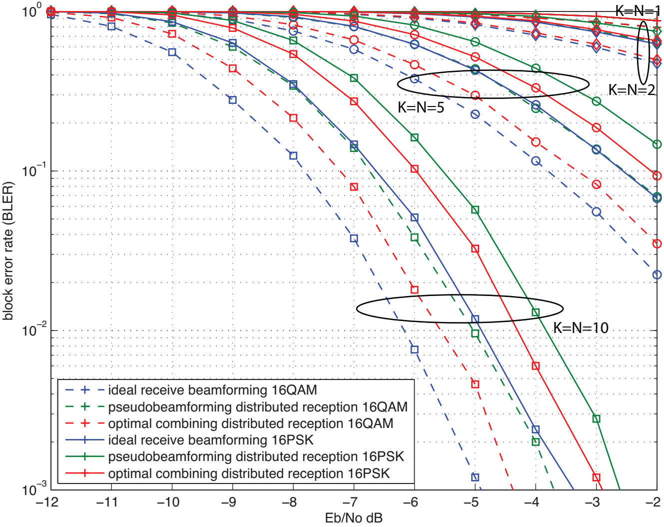
Block Error Rate Performance: 4-PAM/16-QAM



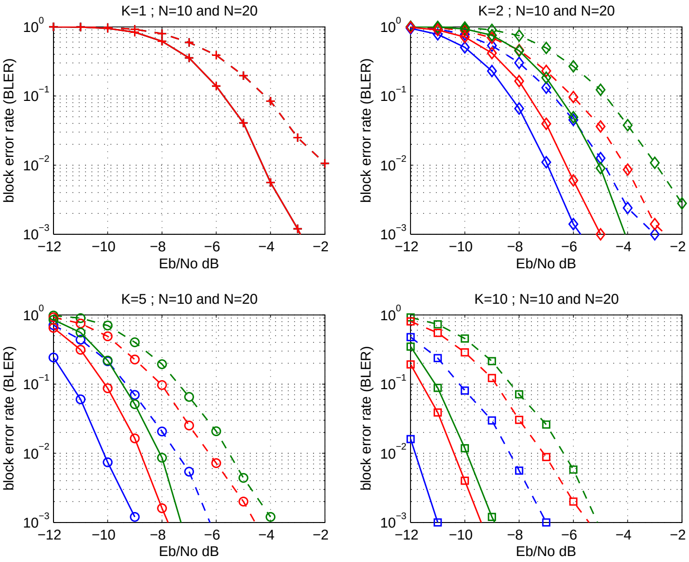
Block Error Rate Performance: 4-PAM vs. QPSK



Block Error Rate Performance: 16-QAM vs. 16-PSK



BLER Performance: BPSK/QPSK Partial Participation



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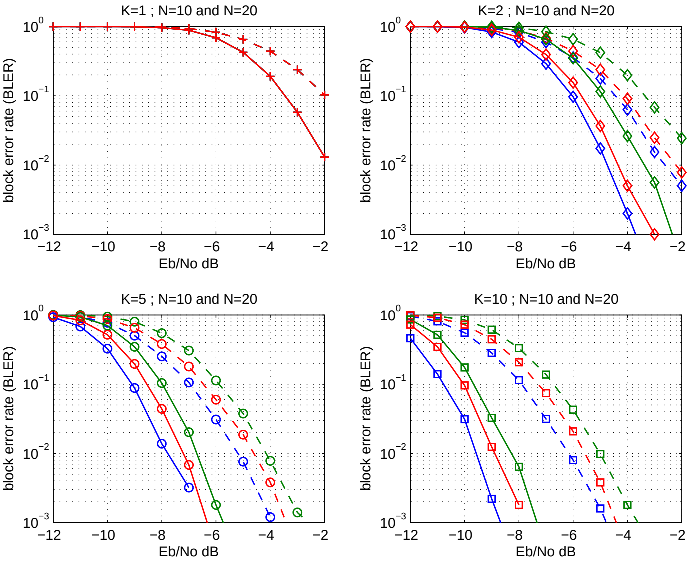

Conclusions

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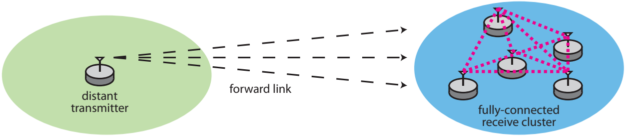
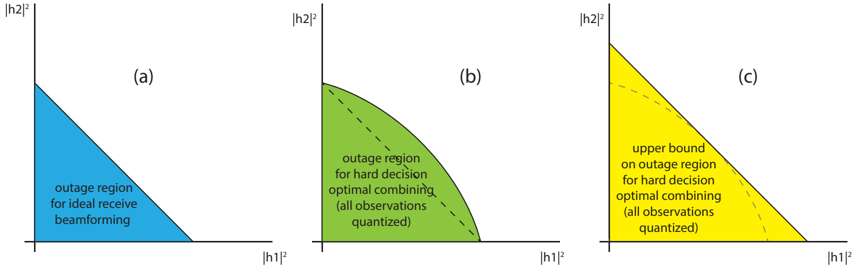


Distributed reception with hard decisions exchanged over a conventional LAN

- Practical, implementable with off-the-shelf network hardware
- No loss of rate when LAN is on separate radio from forward link
- Asymptotic analysis suggests losses with respect to ideal beamforming will be between 1-2 dB
- Numerical SNR results confirm analysis over practical range of SNRs
- Block error rate results show the loss of optimal combining can be less than 1 dB for higher-order constellations.

Bonus: Outage Regions for Distributed Reception

Two-receiver case:



(a) For ideal receive beamforming with Gaussian channel inputs, we have $Y = \|\boldsymbol{h}\|^2 X + W$ and $I_{\boldsymbol{h}}(X;Y) = \log(1 + \|\boldsymbol{h}\|^2)$.

(b) For binary channel inputs and both receivers making hard decisions, we have a 2-input, 4-output DMC and it is possible to numerically compute the outage region $J_{\boldsymbol{h}}(X;Y) < R$.

(c) To show the yellow region is indeed an upper bound, we need to show that $f(\boldsymbol{h}) = J_{\boldsymbol{h}}(X;Y)$ is Shur-convex over $[|h_1|^2, |h_2|^2]$ on $\mathbb{R}^+ \times \mathbb{R}^+$.

Bonus: Showing Shur Convexity (Help!)

Let $x = |h_1|^2$, $y = |h_2|^2$ and $J_{\boldsymbol{h}}(X;Y) = \phi(x,y)$. We need to show

$$(x - y)\left[\frac{\partial}{\partial x}\phi(x,y) - \frac{\partial}{\partial y}\phi(x,y)\right] \geq 0$$

for all $[x,y] \in \mathbb{R}^+ \times \mathbb{R}^+$. We've done the math and simplified things as much as possible. It boils down to showing

$$s(x,y) = \frac{g(x,y)}{g(y,x)} = \frac{xe^{x^2}\left[\ln\left(\frac{1+\text{erf}(y)}{1-\text{erf}(y)}\right) - \text{erf}(x)\ln\left(\frac{1+\text{erf}(x)\text{erf}(y)}{1-\text{erf}(x)\text{erf}(y)}\right)\right]}{ye^{y^2}\left[\ln\left(\frac{1+\text{erf}(x)}{1-\text{erf}(x)}\right) - \text{erf}(y)\ln\left(\frac{1+\text{erf}(x)\text{erf}(y)}{1-\text{erf}(x)\text{erf}(y)}\right)\right]} < 1$$

is true for $0 < y < x < \infty$ with $\text{erf}(a) = \frac{2}{\sqrt{\pi}}\int_0^a e^{-t^2}\,dt$ as the usual error function.

It is easy enough to test this inequality with randomly generated values for x and y, but I have been unable to analytically confirm it is true.